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Thèse Doctorale

Energy and Money in New Frameworks for Macro-dynamics

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La thèse comporte quatre articles écrits en anglais :

- The Effects of Oil Price Shocks in a New-Keynesian Framework with Capital Accumulation (Acurio-Vasconez, Giraud, Mc Isaac et Pham), publié en 2015 à la revue *Journal of Energy Policy*, 86(C), 844-854 ;
- Testing Goodwin with a Stochastic Differential Approach–The United States (1948-2015) (Mc Isaac) à soumettre en fin 2016 ;
- Coping with the Collapse: A Stock-Flow Consistent Monetary Macrodynamics of Global Warming (Giraud, Mc Isaac, Bovari et Zatsepina) soumis en août 2016 à la revue *Ecological Economics* et en cours de révision ;
- Minskyan Classical Growth Cycles: Stability Analysis of a Stock-Flow Consistent Macrodynamic Model (Mc Isaac, Fabre et Bastidas) à soumettre.

Le premier chapitre présente un aperçu des principaux développements de la thèse. Il fait un bilan des liens entre la macroéconomie et l'énergie dans la littérature et montre comment la thèse s'insère dans le débat. Dans un second temps, le premier chapitre démontre la motivation de donner un nouveau carde de modélisation macroéconomique et introduit les autres chapitres. Les quatre chapitres suivants correspondent aux quatre articles.

This thesis contains four articles written in English:

- The Effects of Oil Price Shocks in a New-Keynesian Framework with Capital Accumulation (Acurio-Vásconez, Giraud, McIsaac, and Pham) published in Journal of Energy Policy, 86(C), 844-854;
- Testing Goodwin with a Stochastic Differential Approach–The United States (1948-2015) (Mc Isaac) to be submited by the end of 2016;
- Coping with the Collapse: A Stock-Flow Consistent Monetary Macrodynamics of Global Warming (Giraud, Mc Isaac, Bovari, and Zatsepina) submited in august, 2016 to *Ecological Economics* and under revision;
- Minskyan Classical Growth Cycles: Stability Analysis of a Stock-Flow Consistent Macrodynamic Model (Mc Isaac, Fabre, and Bastidas) to be submited.

The first chapter presents an overview of the thesis' developments. First, the introduction summarizes the state of the art of the linkage between macroeconomics and energy. Second, the introduction motivates the reason why the thesis has changed its modelisation framework and introduces the other chapters. The next four chapters are the four articles.

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Résumé

Energy and Money in New Frameworks for Macro-dynamics

par Florent Mc ISAAC

Depuis la stagflation observée consécutivement à la forte hausse du prix du pétrole en 1973 et 1979, les chocs pétroliers sont considérés comme l'une des sources de fluctuations potentiellement les plus importantes aux États-Unis comme dans de nombreux pays industrialisés. De nombreux articles ont étudié le rôle des chocs pétroliers dans la fluctuation des principales variables macroéconomiques à savoir, la croissance, le chômage, l'inflation et les salaires. Cependant, ces travaux n'ont pas encore permis d'aboutir à un consensus. Le débat s'est même intensifié au cours de cette dernière décennie, en raison d'une absence de réaction forte de l'économie réelle pendant la période d'augmentation du prix du pétrole entre 2002 et 2007. En effet, la récession qu'aurait dû engendrer une telle hausse des prix ne fut observée qu'au moment de la crise des *subprimes* en 2008. Plusieurs hypothèses furent avancées pour expliquer la différence entre les crises des années 1970 et 2000. Blanchard & Gali (2009) et Blanchard & Riggi (2013) évoquent, par exemple, la réduction de la quantité de pétrole utilisée dans la production, la plus grande flexibilité des salaires réels et une meilleure crédibilité de la politique monétaire. Hamilton (2009) et Kilian (2008) suggèrent quant à eux de l'expliquer par l'origine différente des deux chocs pétroliers : un choc d'offre pendant les années 70 et un choc de demande pendant les années 2000.

L'objectif original de la thèse était de réexaminer l'impact des chocs pétroliers sur l'économie réelle par le canal de la dette. Dans un premier temps, sur la base des travaux de Blanchard & Galí, nous proposons un nouveau modèle dynamique d'équilibre général stochastique (DSGE), qui intègre le pétrole à la fois comme facteur de production et comme bien de consommation. En relâchant plusieurs hypothèses adoptées dans Blanchard & Galí, notamment en découplant l'élasticité du PIB vis-à-vis du pétrole avec la part du pétrole dans les coûts de production, ce travail a permis de montrer que l'intensité du pétrole dans la production est encore aujourd'hui une variable fondamentale de la croissance américaine. Aussi, nous montrons que l'efficacité énergétique est un canal déterminant dans l'explication de la diminution de l'impact macroéconomique de la hausse du prix du pétrole. Le troisième facteur qui pourrait expliquer la différence d'impact de l'augmentation des prix du pétrole entre les années 1970 et 2000 serait que les coûts supplémentaires soient absorbés par la dette elle-même grâce à des taux d'intérêts directeurs particulièrement bas. Or, la thèse a permis de mettre en lumière les difficultés du cadre de modélisation DSGE à répliquer l'environnement macroéconomique à l'aube de la crise financière.

Fort de ce constat, je me suis orienté vers le développement d'un nouvel axe de recherche afin de représenter les mécanismes économiques sous un angle différent. Ce nouveau cadre de modélisation met la dette privée au centre de l'analyse macroéconomique et propose une vision alternative sur la crise financière des années 2000. Le formalisme mathématique initial de cette nouvelle perspective est donné par Keen. Ce dernier a formalisé les intuitions de Hyman Minsky qui, dans les années 1970, cherchait à savoir si une nouvelle crise de l'ampleur de celle de 1929 était encore possible. L'avantage premier de ce nouveau cadre de modélisation est qu'il est possible d'y reproduire de façon endogène l'environnement de la crise des *subprimes*. Dès lors, il est possible de faire des recommandations de politiques publiques qui permettent d'éviter ce que l'on a l'habitude d'appeler un “ cygne noir ” ou la réalisation d'une “ queue de distribution ” (probabilité faible de survenir). Ce changement de paradigme n'est pas anodin car il suggère de construire de nouveaux outils pour la modélisation macroéconomique. Si le cadre de modélisation DSGE est très bien développé dans la littérature académique, l'étude de ce nouveau nouvel environnement de modélisation est encore embryonnaire.

La suite de la thèse est articulée en trois articles. Le premier développe des outils d'estimation adaptés au cadre de modélisation retenu. Il permet d'estimer un système multidimensionnel continu dans un environnement macroéconomique où les données sont de faible fréquence (trimestrielle). Le second article généralise la fonction de production de ce nouveau cadre de modélisation et étudie les propriétés dynamiques inhérentes à cette généralisation. Le dernier papier de la thèse calibre ce nouvel environnement macroéconomique au niveau mondial et détaille les effets du changement climatique sur la macroéconomie. Bien que le scénario le plus probable montre un effondrement de l'économie (engendré notamment par la sphère financière), il démontre que si l'action publique est assez forte, l'effondrement peut encore être évité à condition que la transition énergétique soit impulsée très rapidement.

Les conclusions obtenues dans la thèse apparaissent particulièrement importantes dans la mesure où elles livrent les fondations de nouvelles perspectives en modélisation macroéconomique. Ces travaux permettent notamment de mettre en lumière des situations que bon nombre de modèles ne sont pas capables de répliquer comme notamment la crise de surendettement. Dès lors, ce cadre de modélisation peut apporter un éclairage tout autre sur les recommandations en politiques publiques apportées par les modèles usuels.

Le développement de ces travaux entamés dans la thèse pourra aboutir à un cadre alternatif de modélisation décisif pour l'intelligence de la macroéconomie. Il devrait permettre une meilleure compréhension de l'évaluation des relations réciproques entre la sphère financière, la réalité des cycles macroéconomiques réels, l'énergie eet le climat dans ce qui est sans aucun doute l'enjeu de notre génération : la transition écologique.

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Summary

Energy and Money in New Frameworks for Macro-dynamics

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Ever since the stagflation that followed the oil price run-ups of 1973 and 1979, oil price shocks have been considered one of the most influential sources of economic fluctuation in the United States and other developed countries. A large body of literature has analyzed oil price shocks as sources of variation for leading macroeconomic variables such as GDP growth, unemployment rate, inflation, and wages. However, scholars have yet to reach a consensus as to the true impact of oil shocks on the macroeconomic environment. Furthermore, the last decade has seen the debate intensify as the results of the relatively (in comparison with the 1970s) muted reaction of the real economy during the 2002-6 oil price run-up. Indeed, the recessionary effect was only observed during the subprime mortgage crisis of 2008-9. Numerous hypotheses have been put forward to explain the difference in impact during the 1970s versus the 2000s. For instance, Blanchard & Gali (2009) and Blanchard & Riggi (2013) evoked the reduction of the quantity of oil used of a unit of production, more flexible real wages, and a better credibility of the monetary policy. Hamilton (2009) and Kilian (2008) pinpointed a difference in the nature of the shock: whereas the oil shocks of the 1970s were driven by supply, that of the 2000s was led by demand.

The original aim of this thesis was to reevaluate the impact of the oil shock in the 2000s through the debt channel. First, based on the work of Banchard & Gali, we proposed a new dynamic stochastic general equilibrium model (DSGE), which includes oil as an input of production as well as a consumption good. By relaxing some of the hypotheses of Blanchard & Gali, especially the decoupling of the output elasticity of oil with the cost-share in the production, our work demonstrated that oil is still a fundamental variable of the GDP in the United States. Furthermore, we found that energy efficiency is a key factor that explains the muted macroeconomic impact of an increase in oil prices. A third line of inquiry that may explain the difference between the shocks of the 1970s and the 2000s considers the extra costs implied by a higher price of oil that were absorbed by private debt (which was itself exacerbated by low interest rates set by the Federal Reserve in the 2000s). However, we found that DSGE modeling is unable to replicate the macroeconomic environment that led to the subprime mortgage crisis.

In light of these considerations, I reoriented my thesis along the lines of a new angle of research that seeks to represent economic mechanisms differently. Under this new framework, private debt is at the core of macroeconomic analysis. It provides an alternative view of the financial crisis that occurred in the 2000s. The mathematical formalism is provided by Steve Keen, who formalized basic features of Hyman Minsky's insights. During the 1970s, Minsky sought to analyze the likelihood that a new financial crisis equivalent in magnitude to that of 1929 would be possible. The primary advantage of his framework is its ability to reproduce a financial crisis, such as the subprime mortgage crisis, endogenously. As a result, it is possible to provide public policy recommendations that prevent what we call a "black swan," or a realization of the tail of a probability distribution. Such a paradigm shift is not without consequence. It necessitates the development of new tools for macroeconomic modeling. Indeed, while DSGE framework is well developed in the academic literature, the study of this new modeling environment is still germinal.

The resulting thesis, which seeks to develop this work, is composed of three articles. The first develops estimation tools suitable for the new framework. It enables the estimation of a multidimensional continuous system in a macroeconomic environment where data is at a low frequency (quarterly). The second article generalizes the production function of the new framework and studies the dynamical properties inherent to this generalization. The last paper calibrates this new macroeconomic environment at a global scale and delineates the effects of climate change on the macroeconomy. Although the most likely scenario would be an economic collapse induced by the financial sphere, we show that if the public policy is strong enough, the collapse can still be avoided provided that the energy shift be enacted swiftly.

The conclusions of this thesis demonstrate great potential for providing foundations for new perspectives in macroeconomic modeling. The papers included in the thesis allow, in particular, for a better understanding of situations that most macroeconomic models are not able to cope with, including the overindebtedness crisis. As a result, the framework introduced here may provide an alternative and improved perspective for public policy. Further development of the research presented in this thesis may lead to the improvement of other frameworks in the field of macroeconomics. This would allow for a better understanding of complex interactions between the financial sphere, real business cycles, energy, and climate in what is certainly the biggest challenge of our generation : the ecological shift.

Chapter 1

Introduction

La thèse a pour origine le questionnement suivant : est-ce qu'un environnement de plus en plus dérégulé financièrement peut faire émerger de nouveaux canaux de transmission " directs " entre l'augmentation des prix de l'énergie et la macroéconomie ? La simultanéité dans les années 2000 de deux événements qui n'ont pas eu lieu dans les années 1970 sont à la genèse de ce questionnement: (i) le changement d'impacts sur les économies de l'augmentation des prix du pétrole ; (ii) l'environnement financier international significativement plus dérégulé. La thèse avait pour ambition de construire un cadre pour modéliser la monnaie et la finance afin d'étudier le possible canal de transmission de la dette comme facteur d'atténuation (court terme) de l'impact de l'augmentation des prix du pétrole dans la macroéconomie. Le développement de la thèse conduit à questionner le cadre conventionnel de modélisation macroéconomique existant – les modèles DSGE – et va ensuite poser de nouveaux fondements pour la macrodynamique afin de rendre l'étude de tels canaux possible.

1.1 Les cycles économiques et l'énergie : l'historique

L'étude des liens entre l'économie de l'après-guerre et le pétrole est devenue un axe de recherche à part entière en macroéconomie depuis l'échec des négociations avec les majors pétroliers, en 1973, et l'embargo des pays membres de l'OPEP (Organisation des pays exportateurs de pétrole) suite au déclenchement de la guerre du Kippour. A cette époque, l'OPEP a réduit sa production de 5% par mois, conduisant à plus d'un doublement des prix réels du baril de pétrole (qui sont de 20 dollars courants en 1973 et avoisinent les 50 dollars courants en 1974). Les conséquences économiques de cet embargo ont déstabilisé l'économie mondiale. Aux Etats-Unis, l'inflation de l'indice des prix à la consommation passe de 6,3 % en 1973 à 11 % l'année suivante. Le PIB américain a décru de 0.7 % entre 1973 et 1975, avec une baisse des salaires réels de 2.7 % entre 1973 et 1974. Quelques années plus tard, la révolution iranienne de 1979 sera l'événement déclencheur de ce que l'histoire retiendra comme le second choc pétrolier. Depuis lors, de nombreuses recherches ont été menées essayant de mettre en évidence les connexions – devenues évidentes – qui existent entre les variables macroéconomiques et l'énergie.

Avec le recul des données statistiques, une étude par Hamilton (1983) évalue quantitativement les connexions entre les variables macroéconomiques et l'énergie. Par soucis de synthèse, comme dans l'article d'Hamilton, le prix de l'énergie se résume au prix du pétrole. Cette simplification est la conséquence du rôle prépondérant du pétrole dans la matrice énergétique mondiale et par ses qualités intrinsèques en tant qu'énergie : (i) dense, elle offre une grande quantité d'énergie pour un faible volume ; (ii) liquide, elle est facile à pomper, à stocker, à transporter et à utiliser. Le premier argument avancé par Hamilton pour démontrer le caractère non-fallacieux de la relation quantitative entre le pétrole et la macroéconomie est que sept des huit récessions qu'ont connues les Etats-Unis depuis l'après-guerre ont été précédées, durant trois à quatre trimestres, d'une augmentation subite du prix du pétrole. Les travaux d'Hamilton reposent sur le modèle statistique le plus pointu de l'époque en macroéconomie : le modèle macroéconométrique SVAR (*structural vector autoregressive*) de Sims (1980). Hamilton a démontré la relation statistique de causalité de la variation des prix du pétrole vers la croissance économique. Cette première étude pose un premier jalon fondamental dans l'analyse des liens inhérents entre les cycles économiques et l'énergie.

Des recherches plus récentes démontrent le caractère évolutif – changement structurel – de la relation causale. Tout d'abord Mork (1989) étend les travaux d'Hamilton en introduisant une non linéarité. Dans un article de cinq pages, Mork montre qu'en coupant la série des rendements du pétrole en deux, une partie négative et une partie positive, les connexions entre l'énergie et le PIB sont asymétriques. Une hausse du prix du pétrole va avoir une influence négative plus prononcée sur le PIB qu'aurait, au contraire, une influence positive d'une baisse des prix. Néanmoins, Hooker (1996) montre que la relation entre le prix du pétrole et le PIB disparaît dès lors que l'on ajoute les nouvelles données statistiques aux modèles économétriques. En réponse, Hamilton (1996) reconnaît dans un premier temps, dans la même revue, la pertinence et la robustesse des travaux de Hooker, et, dans un second temps, démontre qu'une nouvelle non linéarité entre le prix du pétrole et les cycles économiques peut réconcilier les relations passées et contemporaines de l'époque en considérant la série chronologique suivante : la variation du prix du pétrole à un instant t par rapport à sa valeur maximale au cours des quatre trimestres précédents si tant est qu'elle soit positive. Hamilton justifie cet indicateur par des considérations comportementales qui sont basées sur la théorie économique. Dès lors que le prix du pétrole atteint un pic (relativement à une période récente), les carnets de commandes se contractent et l'économie rentre en récession quelques trimestres plus tard. Hamilton (2003) montre que le lien est plus robuste en utilisant une non linéarité qui considère les trois dernières années plutôt que la toute dernière. Au-delà du débat sur les meilleures données statistiques à considérer pour traiter la question du lien entre le prix du pétrole et le PIB, Hamilton (2011) et Kilian and Vigfusson (2011b), Kilian and Vigfusson (2011a), Kilian and Vigfusson (2013), Kilian and Vigfusson (2014) n'ont cessé de renforcer la robustesse de leurs résultats.

1.2 L'explication des canaux de transmission du choc pétrolier des années 2000

“ *Tous les chocs pétroliers ne sont pas les mêmes* ”, le titre de l'article de Kilian (2009) dans l'*American Economic Review* renvoie au débat économique qui subsiste autour des relations entre l'énergie et le PIB. Il se focalise aujourd'hui sur la simultanéité entre la crise économique de 2007-2008 et l'augmentation continue du prix du pétrole allant de 19,69 dollars courants en 2002 jusqu'à 74,4 dollars en 2006, voire même 134 dollars en 2008. Le graphique 1.1 montre l'évolution du prix réel du pétrole aux Etats-Unis entre 1980 et 2016 en base 1 la première année. Notons que la première donnée, 1980, représente le pic de prix lié à la crise de 1979 qui a eu lieu en fin d'année. On peut observer que les niveaux de la période 2002-2006 tendent à devenir comparables avec l'environnement après le second choc pétrolier de 1979. Le prix atteint un record historique lors de la crise de 2008 quand le pétrole était considéré comme une valeur refuge sur les marchés financiers.

En économie, plusieurs outils de modélisations sont utilisés afin de traiter une question donnée. Deux grandes catégories, avec une intersection non vide, peuvent résumer ces outils: (i) les modèles statistiques ; (ii) les modèles théoriques. Afin d'illustrer la première catégorie, prenons l'exemple du modèle VAR, qui est un cas particulier du modèle SVAR mentionné *supra*. Ce modèle lie les relations statistiques linéaires et des variables macroéconomiques sélectionnées entre les valeurs passées et présentes avec des erreurs gaussiennes. Ce modèle purement statistique teste si une variable permet d'expliquer le comportement d'une autre variable dans un pur cadre statistique (sans hypothèse économique). La seconde catégorie résume notamment les modèles basés sur la théorie économique. Depuis la parution du livre de Keynes (1936) *Théorie générale de l'emploi, de l'intérêt et de la monnaie*, la profession d'économiste est divisée en deux

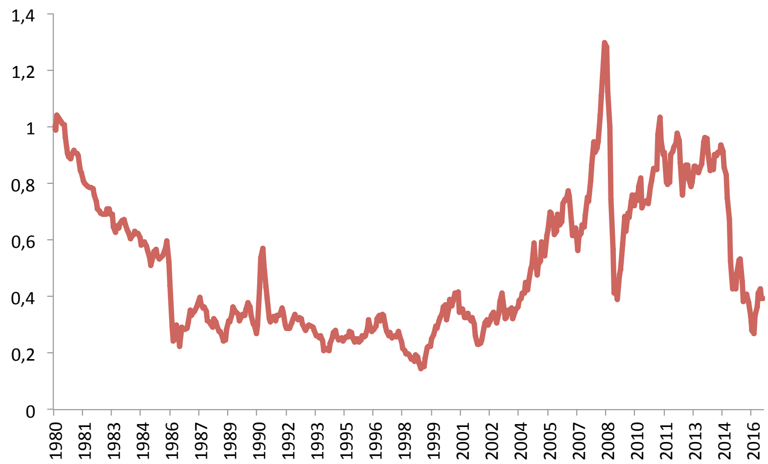

FIGURE 1.1: Prix réels du pétrole – 1980 - 2016. Série normalisée à 1 en 1980.

Source : FRED.

composantes : (i) la microéconomie qui étudie le comportement des ménages et des entreprises ; (ii) la macroéconomie qui considère les grands agrégats globaux de l'économie.

La microéconomie est basée sur un modèle où les consommateurs maximisent leurs propres utilités, les firmes maximisent leurs propres profits, et un système de marché qui rend compte de l'équilibre entre les deux forces en égalisant l'offre et la demande sur tous les marchés. La macroéconomie, elle, était basée sur les interprétations mathématisées des travaux de Keynes sur l'explication de la Grande Dépression. A noter que le travail de mise en équation n'a pas été accompli par John Maynard Keynes lui-même, mais par son contemporain John Hicks.

Dans les années 1960, Lucas et ses contemporains de la nouvelle économie classique ont développé une théorie macroéconomie qui est directement dérivée de la théorie microéconomique standard. Quelques décennies plus tard, le développement de la macroéconomie micro-fondée a amené la science économique à se doter de modèles à base mathématique complexe, connus aujourd'hui sous le nom de : “ *Dynamic Stochastic General Equilibrium* ”, ou modèle DSGE.

Les premiers modèles, connus sous de nom de “ *Real Business Cycle* ” (RBC), font l'hypothèse qu'aucune rigidité ne gène les mécanismes de marché mais aussi que le chômage est volontaire. Sur cette base, les modèles DSGE ajoutent des imperfections de marché de telles sortes que si un aléa exogène venait à toucher l'économie, le retour à l'équilibre serait ralenti. Ce ralentissement va diminuer la croissance économique et entraîner un taux de chômage involontaire de court terme.

Les modèles DSGE dominent la théorie macroéconomique et donc les politiques économiques à travers le monde (cf. Blanchard (2016)). Les hypothèses fondatrices de la modélisation sont :

- Un ménage représentatif maximise son utilité inter-temporelle jusqu'à l'infini au travers de deux composantes : sa consommation et son loisir (ce dernier déterminant l'offre de travail) ;
- Les entreprises dont les managers vont faire les choix d'investissement et d'utilisation des facteurs de production ;
- Les obligations gouvernementales sont achetées par les ménages ;
- Les revenus sont composés par les salaires, les profits et le service de la dette.
- Les entrepreneurs maximisent les flux de profits futurs actualisés jusqu'à l'infini, et donc fixent les prix ; et enfin
- Une banque centrale qui détermine le taux d'intérêt directeur. Ce taux va influencer les comportements de consommation et d'investissement afin de conserver l'inflation et un PIB proche de leurs cibles respectives.

Outre ce cadre, un modèle DSGE standard de la modélisation de l'économie des états-Unis est fait de deux types d'entreprises (les producteurs de biens finaux, qui opèrent sur un marché avec une concurrence pure et parfaite, et les producteurs de biens intermédiaires sur un marché imparfait par un schéma de concurrence monopolistique entre firmes produisant des biens imparfaitement substituables, qui a été développé par Dixit and Stiglitz (1977)) ; une viscosité dans la fréquence de réajustement des prix par Calvo (1983) ; un type de ménage ; un syndicat qui négocie sur le marché des salaires ; et un gouvernement qui a un financement contraint et qui est actif dans la mise en place du taux d'intérêt directeur par la règle de Taylor.

Notons qu'il n'est pas possible de déterminer laquelle de toutes ces approches – empirique ou théorique – comportent le moins d'hypothèses et donc laquelle est la plus “ athéorique ”. En effet, la sélection des données statistiques qui composent une étude est en-soi une hypothèse qui va conditionner les conclusions de l'étude (cf. le débat ci-dessus sur la non linéarité à considérer pour les prix du pétrole). A noter que les études précédemment citées de Kilian utilisent une approche mixte par les SVAR alors qu'Hamilton démontre ses arguments en utilisant les deux approches séparément. Cette modélisation est issue de ce qu'on appelle la “ nouvelle synthèse néoclassique ” (cf. Goodfriend and King (1998)).

Deux études au minimum ont tenté d'analyser les différences d'impacts sur la macroéconomie des chocs pétroliers entre les années 1970 et 2000 en appliquant ce cadre : Blanchard and Galí (2009) et Blanchard and Riggi (2013). En utilisant dans un premier temps un SVAR sur plusieurs pays, les auteurs démontrent statistiquement que les impacts d'un choc des prix du pétrole sur l'économie des états-Unis sont plus modérés dans les années 2000 qu'à la fin des Trente Glorieuses. Dans Blanchard and Galí (2009), cet argument est illustré par un modèle DSGE calibré qui tente d'expliquer la différence entre les deux épisodes de la façon suivante : (i) une moindre part du pétrole dans la production ; (ii) un marché du travail plus flexible ; (iii) une plus grande crédibilité de la politique monétaire. Dans leur étude, ils remettent en question l'influence des fluctuations du prix du pétrole sur l'économie des années 2000. Dans un modèle analogue, Blanchard and Riggi (2013) ont calibré certains paramètres du modèle DSGE à l'aide de techniques de minimisation d'erreurs et confirment les trois dernier facteurs explicatifs.

Le premier travail de la thèse[1] a été conduite à partir de l'analyse faite par Blanchard and Galí (2009) en affinant son cadre pour mieux appréhender l'énergie et les effets de la financiarisation de l'économie. Une première étape a consisté à ajouter l'accumulation du capital afin d'augmenter le réalisme du modèle DSGE de Blanchard and Galí (2009). Cet ajout permet d'identifier un élément clé permettant de comprendre le changement d'impact d'une augmentation du prix du pétrole sur l'économie permet de rendre compte l'efficacité énergétique du secteur industriel a fortement changé dans les années 1980. Même si l'efficacité énergétique est prise en compte dans le modèle de Blanchard and Galí (2009) par la part des coûts du pétrole dans la production, nous montrons dans le premier article de la thèse que ces deux variables doivent être découplées.

Un résultat classique de la maximisation du profit – dans un cadre de concurrence pure et parfaite, avec une fonction de production a rendement constant, et sous l'hypothèse qu'aucune variable est omise – est que l'élasticité d'un facteur de production est égal, à l'équilibre, à sa part des coûts dans la production – hypothèse utilisée dans les deux dernières études citées. Or, comme le démontre Kümmel et al. (2008), ce résultat est remis en cause dès lors que l'on ajoute des contraintes saturées de production qui font apparaître des coûts fictifs. Ces derniers, dans un cadre d'équilibre, cassent la relation qui égalise l'élasticité avec le coût du facteur de production. L'équilibre vérifie la relation

$$\varepsilon_i = \frac{x_i\left(p_i - \lambda\frac{\partial f(x)}{\partial x_i}\right)}{\sum_{i=1}^{n} p_i x_i - \lambda x_i \frac{\partial f(x)}{\partial x_i}},$$

[1] Ce travail est commun entre Véronica Acurio, Gaël Giraud et Ngoc-Sang Pham a été publié depuis 2015 dans la revue *Energy Policy* : Acurio V. et al. (2015).

avec ε_i représentant l'élasticité de substitution, p_i le prix de l'intrant i, x_i la quantité de l'intrant i, λ le prix fictif, et f est une contrainte saturée. Cette dernière permet de capturer des phénomènes comme la restriction géologique des ressources fossiles, géopolitiques, climatiques, syndicales, institutionnelles, etc. A noter que si le coût fictif n'existe pas (i.e. $\lambda = 0$), l'égalité entre la part du coût de production et l'élasticité de substitution est vérifiée. Néanmoins, la présence d'au moins une contrainte – qui est très fort probable, voire certain – donne lieu a un découplage entre l'élasticité de substitution et la part des coût dans la production.

Une étude complémentaire par Giraud and Kahraman (2015) fait une analyse empirique de l'élasticité de substitution de l'énergie primaire dans la production de 33 pays. L'estimation de cette élasticité se trouve entre 40% (en France) et 70% (aux États-Unis) avec une moyenne autour de 60%. Cette découverte montre que la méthode de la part des coûts dans la production aurait sous-estimé d'environ 6 fois cette élasticité.

Ce découplage a permis de réexaminer, dans un cadre DSGE, l'impact de l'énergie dans la production et a guidé l'article vers les conclusions suivantes : (i) L'élasticité de substitution du pétrole estimée avec les techniques Bayésiennes est bien plus haute – dans le s proportions de Giraud and Kahraman (2015) – que ce qu'un coût des facteurs de production pourrait suggérer ; (ii) l'efficacité énergétique est un canal déterminant de l'explication de la diminution de l'impact macroéconomique de la hausse du prix du pétrole ; (iii) l'intensité du pétrole dans la production est encore aujourd'hui une variable fondamentale de la croissance américaine.

1.3 Modéliser la financiarisation de l'économie pour expliquer la crise financière

Pour rappel la thèse a pour ambition de construire un cadre pour modéliser la monnaie et la finance afin d'étudier le canal possible de transmission de la dette comme facteur d'atténuation (court terme) de l'impact de l'augmentation des prix du pétrole dans la macroéconomie. Ce canal de transmission a été bien résumé par Stiglitz (2015) dans son livre *La grande fracture* paru au cours de l'élaboration de la thèse :

"*La guerre en Irak a aggravé les choses en provoquant une très forte hausse des cours du brut* [de pétrole]. *Puisque l'Amérique est très dépendante des importations de pétrole, nous avons dû leur consacrer plusieurs centaines de milliards de plus – des fonds qui, sans la hausse de cours, auraient servi à acheter des produits américains. Normalement, cela aurait dû provoquer un ralentissement de l'économie, comme dans les années 1970. Mais, face à ce défi, la Federal Reserve a eu la réaction la plus court-termiste qu'on* *puisse imaginer. Une marée de liquidités a envahi les marchés des prêts hypothécaires : tant d'argent était disponible qu'il y en avait même pour ceux qui, normalement, n'aurait pas été en position d'emprunter*". (Stiglitz (2015), p.44).

La crise des subprimes est avant tout une crise de l'endettement privé. Comme le rappelle Stigliz, pendant les années 2000 alors que les prix du pétrole suivaient une augmentation continue (et stationnaire), beaucoup de ménages, même les plus modestes, ont pu emprunter avec, pour ainsi dire, aucun collatéral (cf. Geanakoplos (2009)). Ces emprunts bancaires encouragés par les taux directeur bas de la FED – allant de 0,98 % à 5,3 % entre 2002 et 2006 – permettaient à tous les Américains de vivre à crédit, et par conséquent, de ne pas subir les effets à court terme de l'augmentation des prix du pétrole. Lorsque la thèse a été initiée, le cadre conventionnel DSGE avait été retenu comme outil de modélisation. Après avoir ajouté l'énergie dans le processus de production d'un modèle DSGE complet avec travail et capital productifs, l'étape suivante était de modéliser les banques, et donc la dette, dans le but de répliquer l'environnement économique des années 2000. Pour mettre en place la modélisation du canal de transmission financière dans les modèle DSGE, la revue de littérature a abouti, entre autres, à l'exploration détaillée de deux articles : Gertler and Karadi (2011) et Eggertsson and Krugman (2012).

Gertler and Karadi (2011), qui s'appuient sur l'article fondateur de Gertler and Kiyotaki (2010) dans l'intégration des systèmes financiers des modèles DSGE, ont analysé les effets des politiques monétaires non conventionnelles mises en place par la FED à la suite de l'éclatement de la bulle financière de 2008. Toutefois, dans cette analyse, l'unique moyen mis en œuvre pour répliquer les causes de la crise financière se situe à travers un choc exogène détériorant la " qualité " des actifs. Cette méthodologie nous suggère donc que la crise financière, qui, en réalité, est la conséquence d'un surendettement endogène non contrôlé des agents économiques ne pouvant pas honorer leur dette, est une " surprise absolue ", un " cygne noir ", un événement extrême qui résulte de la réalisation d'un événement de queue de distribution (cf. Taleb (2007)). Autrement dit, dans les trajectoires suggérées par le modèle, une crise financière ne peut pas subvenir autrement que par des forces qui sont indépendantes du système. En effet, les modèles DSGE étant conçus pour être un outil d'analyse de perturbations stochastiques autour d'un équilibre, un surendettement ne peut résulter que d'une " anomalie " du système. Pour illustration, en utilisant un modèle DSGE, bon nombre d'institutions dont l'OCDE a prévu que 2008 allait être une année exceptionnelle en terme de croissance économie.

Néanmoins, la crise financière des années 2000 a permis de remettre en lumière la recherche des travaux d'un économiste " oublié " : Hyman Minsky. Ses travaux sur l'hypothèse de l'instabilité financière ont trouvé un écho important lorsque le journal

The New Yorker a publié un article titré : “ *The Minsky Moment* ” le 04 février 2008. Au cours des années 1980, alors que bon nombre d’économistes louaient les vertus de la dérégulation financière, Hyman Minsky, qui cherchait à savoir si une nouvelle Grande Dépression était encore possible, envisageait une probable déstabilisation de l’économie par cette même dérégulation. L’hypothèse d’instabilité financière de Minsky résume le fait que le capitalisme se déséquilibre lui-même de façon intrinsèque.

Afin d’illustrer l’instabilité de l’économie, commençons par considérer une période où la stabilité économique est de plus en plus présente, le capitalisme – par les investissements – encourage la prise de risque et l’optimisme. A leurs tours, la prise de risque et l’optimisme entraînent l’innovation qui transforme simultanément la production et la société. Néanmoins, l’innovation et la croissance génèrent toutes deux de l’incertitude dans l’environnement économique. En utilisant l’argument que, pour les agents économiques, l’appréciation du futur est guidée par l’évaluation du passé, Minsky rend compte de l’esprit moutonnier qui dirige les choix d’investissements. Par conséquent, une période de croissance constante et prolongée pousse le capitalisme à passer d’un état de “ découragement ” à un état d’“ euphorie ”. Une longue période de croissance continue fait augmenter les espérances sur le futur et, ainsi, tend à augmenter les effets de levier financier. Or, dès lors que l’investissement – encouragé par la prise de risque – est supérieur aux profits retenus d’une entreprise, la dette privée vient à augmenter. La période d’euphorie peut alors mener à une crise économique et financière. En effet, cet environnement d’euphorie va encourager les banques à financer plus de projets qui sont, pour certains, destinés à échouer. Des pertes sur les marchés financiers vont donc s’accumuler en période de boom économique ; dans cette même période de boom, la forte demande de financement fait augmenter les taux d’intérêts, réduisant par conséquent la viabilité financière de tous les autres investissements – mêmes ceux qui auraient vu le jour dans une période moins euphorique. Au plus fort de la période de boom, les acteurs sur les marchés financiers vont vendre leurs actifs en réponse à un trop perçu dans l’évaluation (prix) de ces mêmes actifs déclenchant ainsi un effondrement du crédit[2]. De plus, en période de boom économique, comme la dette s’accumule à mesure que l’investissement est en excès des profits non redistribués, l’endettement grimpe ainsi que le poids du paiement du service de la dette. L’augmentation des coûts du service de la dette va faire diminuer les profits escomptés, et va donc faire diminuer le ratio d’investissement. Ainsi, la période de boom laisse place à une période nouvelle de récession. La période de récession fait baisser les taux d’intérêt, les investisseurs seront moins euphoriques, le taux de profit retournera à des niveaux modérés, qui conduira

[2] Eggertsson and Krugman (2012) illustrent ce moment en faisant l’analogie avec le dessin animé *Bip Bip et le Coyote*. En effet, dans ce dessin animé il est habituel de voir le coyote continuer sa course dans le vide peu après avoir atteint le bord d’une falaise. Alors que tout se passait bien, dès lors que le coyote s’aperçoit qu’il court dans le vide, il se met à tomber.

à une période de croissance modérée. L'optimisme revient peu à peu et le schéma se répète mais, cette fois-ci, la nouvelle période d'euphorie démarre avec une dette plus élevée que la précédente. Ce cycle se répète jusqu'à ce que le paiement du service de la dette ait atteint un niveau trop élevé et, qu'en absence de faillite, la dette s'accroisse indéfiniment entraînant une dépression de l'économie. C'est par ce constant que Minsky place dans les années 1970, voire même avant, le rôle de la dette privée au cœur de sa théorie macroéconomique.

Paul Krugman et Gauti Eggertsson ont publié dans le *Quaterly Journal of Economics* l'article : " *Debt, Deleveraging, and the Liquidity Trap: A Fisher-Minsky-Koo approach* " (Eggertsson and Krugman (2012)). Ils tentent de réconcilier les modèles néo-classiques DSGE avec, entre autres, les intuitions de Minsky en y expliquant la dette privée (qui, comme l'explique Stiglitz (2015) ou Giraud (2015), a entraîné les agents économiques dans une illusion financière). La méthodologie est simple : afin d'introduire la dette privée dans le cadre DSGE, le ménage représentatif est scindé en deux : un ménage patient qui prête, et un ménage impatient qui empreinte. Dans ce cadre, et afin de simuler une crise financière à la Minsky, Eggertsson and Krugman (2012) introduisent une variation ad-hoc et non expliquée de la borne supérieure de la dette des ménages impatients. Plus encore, le caractère endogène du crédit monétaire n'est pas pris en compte car le prêt est, dans ce modèle, qu'un simple transfert de pouvoir d'achat d'un agent à un autre : ni les banques ni la monnaie n'existent dans le modèle qu'il a construit.

La conséquence des conclusions tirées par les paragraphes précédents est : les modèles DSGE ne semblent pas être adaptés pour répliquer fidèlement les trajectoires économiques empruntées lors de la crise des *subprimes*. La thèse va donc explorer un autre cadre de modélisation pouvant répliquer des trajectoires permettant les crises de façon endogène et ainsi apporté un éclairage sur les observations initiales de la thèse.

En développant plus avant sur la modélisation DSGE, Blanchard (Blanchard (2016)) publie un *policy paper* ayant pour fil conducteur la démonstration que les modèles DSGE présentent des défauts majeurs. L'argument principal de Blanchard repose sur le fait que les hypothèses de modélisation émises sur le comportement des consommateurs et des entreprises sont profondément erronées par rapport aux observations empiriques. Ceci remet en cause le cadre de modélisation macroéconomique qui repose sur des fondations microéconomiques :

"*Ils sont basés sur des hypothèses peu désirables. Pas seulement simplificatrices, comme tous les modèles doivent faire, mais sur des hypothèses profondément différentes de ce que nous connaissons tous des consommateurs et des entreprises.* " (Blanchard (2016), p.1)

Dans un autre article, Kocherlakota (2016) dans un papier appelé " *Toy Models* " fait l'aveu que la communauté des macroéconomistes issue de l'école de la synthèse néoclassique n'ont pas de théorie macroéconomique réussie gravée dans le marbre et aussi que les choix de modélisation faits depuis 30 ans peuvent être remis en cause à tout moment. Sa critique va même plus loin en affirmant que les modèles basés sur de tels fondations ne contribueront pas à mener vers de réels progrès.

En outre, Romer (2016), connu pour avoir développé une théorie de la croissance endogène, a écrit (p.1) : " *Durant ces trois dernières décennies, les méthodes et les conclusions de la macroéconomie se sont détériorées au point que la majorité des travaux dans cette filière ne peut plus être qualifiée de recherche scientifique.* ". Romer fait bien allusion aux modèles DSGE et donc au cadre de modélisation dominant. La thèse principale que défend Romer dans son article est que les résultats des méthodes statistiques utilisées pour les modèles DSGE – l'inférence Bayésienne – sont d'autant plus biaisés que les hypothèses a priori des modélisateurs sont fausses.

Toutefois, les critiques sur les modèles DSGE ne sont pas neuves. Steeve Keen dans son livre *l'imposture économique* – Keen (2014) – démontre, entre autres, que les cadres de modélisations macroéconomiques issus de l'équilibre général sont méthodologiquement incapables de répliquer la crise financière de 2007-2008 (cf. Chapitre X : *Pourquoi ils n'ont pas vu venir la crise* p.241-306). Plus encore, il remet en cause les concepts structurant ces modélisations : l'offre et la demande agrégées. D'une part l'offre, Keen suggère que la fonction de production ne peut être qu'à rendement croissant (cf. Chapitre III : *Le calcul de l'hédonisme*, p.67-106). D'autre part la demande, Keen rappelle que l'un des initiateurs de la théorie économique mathématisée de l'équilibre général, Gérard Debreu, a démontré lui-même en 1975, que l'individualisme méthodologique sur lequel est fondé la macroéconomie néoclassique est erroné. En effet, le théorème de Sonnenschein–Mantel–Debreu démontre que même si tous les individus étaient rationnels au sens de la théorie microéconomique (que la courbe de demande de chaque individu décroisse en fonction des prix), personne ne peut présager d'un comportement rationnel au niveau *macro*. En d'autres termes, un phénomène d'émergence a lieu : tout peut advenir à l'échelle agrégée (cf. Chapitre IV : *Le prix de tout et la valeur de rien*, p.139-165).

La deuxième partie de la thèse sera consacrée au développement du programme de recherche sur la modélisation Minskyienne de l'économie. D'abord, quel cadre peut répondre de la façon la plus vraisemblable à la question initiale ? Un début de réponse vient à la lecture de troisième partie du livre de Keen : *les alternatives* (p.366-500).

Keen, lui-même, propose dans son livre un cadre de modélisation inspiré de la logique Minskyenne. Son modèle se base sur les travaux de Goodwin (1967) capables de reproduire le caractère cyclique de la croissance économique. Le modèle de Goodwin (1967) est basé sur une dynamique non-linéaire proie-prédateur (ou de Lodka-Volterra) entre le taux d'emploi et la part des salaires dans la valeur ajoutée. Le modèle est fondé selon les hypothèses suivantes :

- une productivité du travail croissante ;
- une trajectoire d'augmentation de la population ;
- un ratio constant entre le stock de capital productif et la valeur ajoutée ;
- une courbe de négociation salariale (autrement appelé courbe de Phillips de court terme) ;
- le profit qui égale l'investissement ;
- une équation de l'accumulation du capital ;
- une fonction comportementale de négociation salariale.

A noter que la notion de fonction comportementale dans ce cadre de modélisation est une réponse possible vis-à-vis des phénomènes d'émergences du théorème de Sonnenschein–Mantel–Debreu. En effet, ce modèle macroéconomique ne se repose pas sur une fondation *micro*. Les trajectoires d'un tel modèle forment des cycles endogènes et sont représentées par une orbite fermée entre les variables d'états (cf. Graphique 1.2)–la dynamique va repasser un nombre infini de fois vers sa condition initiale. Quatre phases du cycle économique sont représentées dans les trajectoires :

- une phase de *récupération* (en bas à gauche du Graphique 1.2) : l'accroissement des profits encourage l'investissement et la croissance future pour l'accumulation du capital provoquant l'augmentation de l'emploi.
- une phase de *boom* (en haut à gauche du Graphique 1.2) : la croissance économique fait augmenter le taux d'emploi vers des niveaux assez hauts, ce qui fera donc augmenter les salaires. Par conséquent, le taux de profit diminue entraînant un ralentissement de la croissance économique.
- une phase de *récession* (en haut à droite du Graphique 1.2) : la perte de croissance économique fait diminuer l'emploi, mais pas de façon assez marquée pour diminuer les salaires, réduisant peu à peu les profits et donc l'investissement.
- une phase de *dépression* (en bas à droite du Graphique 1.2) : l'investissement ayant été faible lors de la période précédente, le taux d'emploi continue de diminuer jusqu'à entraîner une diminution des salaires, contribuant ainsi à l'augmentation des profits.

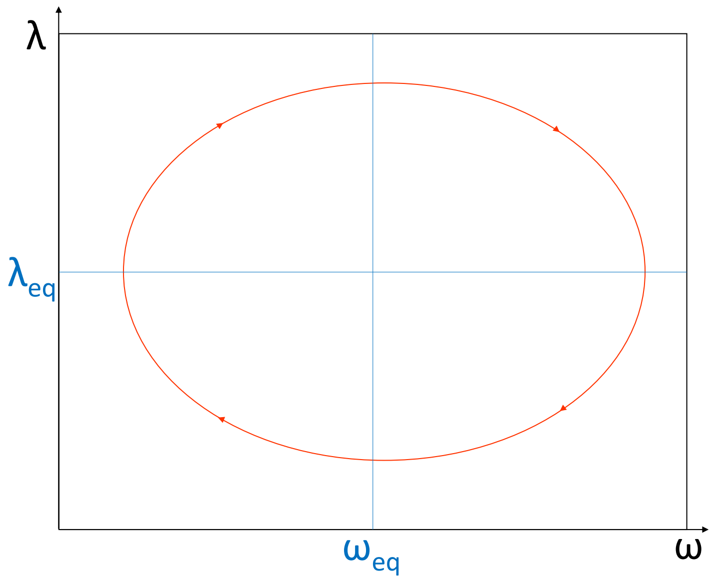

FIGURE 1.2: Le diagramme de phase entre ω, la part des salaires dans la valeur ajoutée, et λ, le taux d'emploi.

En ajoutant une troisième variable d'état au modèle de Goodwin avec la dette privée, Keen parvient à mathématiser les intuitions de Minsky *via* un modèle mathématique où les crises financières sont endogènes. Le plus remarquable, c'est qu'avec ce modèle il ait réussi à simuler des scénarios similaires à la crise financière de 2008, treize ans avant que celle-ci ne survienne (cf. Keen (1995)).

Dans le modèle de Keen (1995), l'ajout d'une troisième variable d'état par la dette rend le système dynamique dissipatif et permet l'existence de plusieurs équilibres. Grasselli and Lima (2012) montrent que le modèle de Keen admet deux équilibres vers lesquels les solutions du système d'équations vont converger. En fonction de la position initiale de l'économie, le modèle suggère que cette même économie peut converger vers un “ bon ” équilibre (où toutes les valeurs qui la composent ont des niveaux finis) ou bien suivre une trajectoire où la dette explose provoquant un effondrement de l'économie. Ces propriétés ont permis à Keen de fournir des trajectoires représentant la Grande modération – caractérisée par une faible volatilité des variables macroéconomiques durant les années 1990 et début des années 2000 –, puis une turbulence macroéconomique annonciatrice d'une crise par surendettement privé.

Passer, ici, du cadre DSGE fondé à partir des travaux de Kydland and Prescott (1982) à celui de Goodwin (1967) qui est au cœur de la dynamique de Keen, n'est pas qu'un changement de paradigme, c'est aussi un changement méthodologique radical. Alors que les modèles DSGE sont omniprésents dans la sphère académique, la littérature qui se développe autour du modèle de Goodwin est beaucoup plus modeste. C'est pour cette raison que la thèse a dû se tourner vers le développement de ce programme de recherche afin de pouvoir se doter des outils nécessaires à la reproduction des conditions macroéconomiques *post* crise de l'endettement.

1.4 Le développement d'un cadre de modélisation

Les trois derniers chapitres de la thèse sont consacrés au développement du cadre de modélisation inspiré de la logique de Lotka-Volterra (*i.e.* proie-prédateur) et des modèles stocks et flux cohérents (où les stocks sont biens la sommes de leurs propres flux).

1.4.1 Les outils statistiques

Le deuxième article de la thèse[3] développe des outils empiriques et de validations de modèles dans ce nouveau cadre de modélisation. En effet, l'un des manquements cruciaux de ce programme de recherche est une méthodologie qui permet d'étudier finement l'inférence statistique des modèles étudiés étant donnée la faible fréquence – trimestrielle au mieux – des données statistiques. Une première tentative, avec des conclusions plutôt mitigées, a été faite par Harvie (2000), qui en estimant le modèle de Goodwin (1967), équation par équation avec des outils conventionnels d'économétie, concluait que ce modèle ne constituait pas nécessairement la meilleure dynamique pour reproduire les trajectoires du taux d'emploi et de la part des salaires dans la valeur ajoutée. Un autre travail utilisant une méthodologie analogue, celui de Grasselli and Maheshwari (2016), conclut sur des résultats plus optimistes quant à la capacité du modèle proie-prédateur de Goodwin à répliquer les séries temporelles observées.

La méthodologie que je propose dans le papier est tout autre. Elle est partiellement inspirée des techniques d'estimation utilisées par les modèles DSGE, sans toutefois tomber dans la critique de Romer (2016) car elle n'est fondée sur aucun a priori sur les valeurs de paramètres. Dans l'article, j'étends la modélisation de Goodwin en introduisant des perturbations stochastiques (en transformant des équations différentielles ordinaires en équations différentielles stochastiques). Cette extension permet d'utiliser les outils statistiques de Durham and Gallant (2002) que j'étends au cadre multidimensionnel et qui peuvent être répliqués pour n'importe quel modèle dans ce cadre pourvu qu'il puisse être écrit de façon réduite. La méthodologie est basée sur la simulation de la fonction de vraisemblance par des ponts Browniens permettant de pallier au manque de données statistiques. Pour illustrer les propos, un pont Brownien simule une trajectoire empruntée entre deux points qui, par une méthode de Monte Carlo (*i.e.* la réplication

[3] L'article a été soumis à *Journal of Economic Dynamics and Control.*

d'un nombre significatif de trajectoire) permet d'affiner l'estimation même en l'absence de plus d'observables.

Outre le modèle de Goodwin, l'extension proposée par Van der Ploeg (1985) est aussi étudiée. Van der Ploeg a étendu le modèle de Goodwin en introduisant une fonction de production CES et relâche par conséquent l'hypothèse inhérente aux hypothèses de Goodwin qui repose sur un ratio constant de la part du capital sur la production. Cette extension n'est pas anodine car elle change considérablement le paysage dynamique du cadre d'analyse en transformant un modèle structurellement instable en un modèle avec des trajectoires asymptotiquement stables, convergent vers un équilibre. A la suite de l'analyse de l'estimation, le papier conclut que le modèle de Goodwin avec une fonction de production CES est plus apte à répliquer les trajectoires passées que le modèle avec une fonction de production Leontief. De plus, la stratégie de *backtesting* – qui teste l'erreur de prévision commise par le modèle étudié relativement à un modèle purement statistique connu pour ses qualités de prévision court terme : le modèle VAR – montre que le modèle de Van der Ploeg donne de meilleures performances en terme de *backtesting* que le modèle avec une fonction de production Leontief et que le modèle VAR lui-même. En d'autres termes, un modèle avec une fonction de production CES est le meilleur pour faire de la prévision " court terme " (inférieure à deux ans). En plus de démontrer la viabilité des modèles utilisées dans ce nouveau cadre de modélisation, l'article donne au programme de recherche des outils nécessaires afin compléter les analyses des modèles.

1.4.2 Des outils quantitatifs et qualitatifs

Dans la littérature basée sur le modèle de Keen, il est usuel de traiter une extension (un schéma de Ponzi, le relâchement de la loi de Say, etc.), en étudiant les équilibres de long terme et en évaluant leur stabilité locale puis en illustrant les dynamiques par des simulations (par exemple Grasselli and Lima (2012) ou Grasselli and Nguyen-Huu (2016)).

Le troisième papier de la thèse[4] propose d'étendre les travaux de Van der Ploeg (1987) et d'étudier le comportement de ce modèle lorsque la dette est ajoutée de façon analogue à Keen. La motivation première est de tester la robustesse des résultats de Keen par une généralisation du modèle au niveau de la technologie de production. Dans un paradigme où la fonction production est une CES, le modèle de Keen devient alors un cas limite obtenu lorsque l'élasticité de substitution entre le capital et le travail est nulle. Néanmoins, un problème apparaît lorsque l'on veut étudier la stabilité locale des équilibres : en plus du fait que les matrices Jacobiennes (les outils qui nous permettent

[4] L'article a été écrit avec Adrien Fabre et Daniel Bastidas Cordoba et a été soumis à xxx.

l'étude de la stabilité des équilibres) ne peuvent être résolues “ simplement ”, certains équilibres eux-mêmes ne peuvent être trouvés analytiquement sans rogner les hypothèses de généralité sur la fonction d'investissement (*i.e.* dès lors que cette fonction est plus complexe qu'une fonction affine, la résolution devient impossible). Face à ce constat, cet article choisit la voix de la simulation numérique afin de parvenir à ses conclusions.

Nous montrons que le modèle de Keen généralisé – permettant une substitution entre le capital et le travail – satisfait les mêmes propriétés mathématiques des équilibres sauf dans le continuum des fonctions de production entre la Cobb-Douglas – avec une élasticité égale à un – et la fonction linéaire – où l'élasticité est infinie. Dans cet ensemble, l'équilibre qui représente l'effondrement n'est plus localement stable. De plus, nous montrons par une étude quantitative que, pour une calibration donnée, par effet substitution, les bassins d'attractions (*i.e.* l'ensemble des points dans lequel le système converge vers un équilibre donné) de l'équilibre économiquement désirable ont un volume substantiellement plus grand que dans le cas limite de Keen. Ceci montre que la substitution facilite la convergence de l'économie vers un équilibre économiquement souhaitable. De plus, les résultats tendent à montrer que, dans le cas CES, les changements quantitatifs du paysage dynamique sont plus faibles que dans le cas Leontief. Ce qui démontre, une fois encore, qu'en cas d'inférence statistique, une modélisation incorporant une fonction CES s'adapterait mieux à cet exercice étant donnée que cet exercice ne donne pas une estimation paramétrique exacte, un modèle moins sensible à ses paramètres est moins assujetti à rendre compte de trajectoire qui dévie très rapidement des trajectoires observés dans le passé.

1.5 Un premier modèle calibré

Néanmoins, une question inhérente à ce cadre de modélisation reste en suspend : peut-on arriver à l'effondrement de l'économie dans un modèle calibré et raisonnable ? En effet, Nguyen-Huu and Pottier (2016) montre, entre autres, que le paysage dynamique est très sensible au choix de la fonction d'investissement. Cette sensibilité se retrouve notamment lors des trajectoires d'effondrement, les simulations numériques peuvent guider certaines variables d'état vers des valeurs économiquement irréalisables ou irréalisées. Cette anomalie est souvent due à la propriété exponentielle de la fonction d'investissement qui, lorsque la simulation amène la solution du système hors du cadre des observations historiques, peut donner des valeurs qui peuvent ne pas être économiquement raisonnables. Le dernier papier de la thèse calibre un modèle de taille moyenne en couplant le modèle de Keen avec une boucle de rétroaction climat. Il démontre que dans une calibration vraisemblable au niveau monde, le paradigme Minskien, qui est le fondement du nouveau cadre macrodynamique utilisé dans la thèse, est capable de montrer qu'un effondrement par le surendettement lié au climat est possible et réaliste par une fonction d'investissement affine.

Ce modèle n'aborde pas l'impact direct de l'énergie sur la croissance, mais les conséquences de son utilisation passée, présente, et future dans le modèle macroéconomique de Keen. La boucle de rétroaction est simple : la production d'aujourd'hui émet du CO_2; le CO_2 s'accumule dans l'atmosphère et augmente le forçage radiatif des gaz à effet de serre. Le forçage radiatif augmente l'anomalie de température qui elle, va dégrader la production future. Le titre de ce chapitre[5] : " Faire face à l'effondrement " fait donc écho à la caractéristique du modèle de Keen qui, contrairement au modèle de Nordhaus (nommé DICE, il couple un modèle climatique avec un modèle macroéconomique basé sur des comportements issue de la théorie de l'équilibre général) dont l'article s'est inspiré pour le module climatique, est capable de rendre compte d'un effondrement de l'économie par la sphère financière. Dans ce chapitre, plusieurs scénarios sont étudiés (et quand c'est possible, de façon croisée):

- une croissance de la productivité du travail de 2% qui représente la calibration du passé ;
- une croissance de la productivité du travail de 1.3% qui reprend les travaux de Gordon (cf. Gordon (2014)) ;
- une croissance de la productivité du travail de 1.5% qui reprend les travaux de Nordhaus (cf. Nordhaus (2014)) ;
- une croissance de la productivité du travail qui dépend de l'augmentation des températures à la Burke et al. (2015) ;
- une croissance de la productivité du travail de type Kaldor-Verdoorn qui représente un scénario de stagnation séculaire (cf. Verdoorn (2002)) ;
- une fonction de dommage à la Nordhaus (2014) ;
- une fonction de dommage à la Weitzman (2012), plus pessimiste que la précédente ;
- une fonction de dommage à la Dietz and Stern (2015), encore plus pessimiste que la précédente.

[5]L'article a été écrit avec Gaël Giraud, Emmanuel Bovati et Ekaterina Zatsepina. Il a été soumis à la revue *Ecological Economics*.

Ces différents scénarios donnent, d'une part, différentes visions sur la croissance de long terme portée par la croissance de la productivité du travail et, d'autre part, différents impacts des effets du changement climatique sur la production. Il s'avère que le scénario le plus réaliste annonce une décroissance du PIB mondial dès la seconde partie du XXIème siècle avec un effondrement total de l'économie au courant du siècle suivant. Afin d'éviter un effondrement, le modèle suggère des prix du carbone similaires à Dietz and Stern (2015) (par tonne de CO_2 en dollars constant 2005) : 74 en 2015 et 306 en 2055. Par rapport à la spécification du modèle, ce prix implique une transition énergétique rapide et terminée peu après 2055.

Ce chapitre se conclut par l'étude des trajectoires des prix du carbone à adopter pour éviter une augmentation de +1.5°C d'anomalie de température par rapport à l'ère préindustrielle. De plus, un test de sensibilité sur le climat est conduit à travers cette étude. Nous montrons que si l'anomalie de température est de +6°C si la concentration de CO_2 est doublée, il est déjà trop tard. En cas d'une augmentation de +2.9°C les prix doivent être, en 2020, à plus de 260 dollars constant 2005 par tonne de CO_2 impliquant une transition énergétique très rapide, voire imminente. Enfin si l'augmentation de température est de +1.5°C, les prix devront être autour de 100 par tonne de CO_2 en dollars constant 2005 à l'horizon 2050. étant donnée l'incertitude des climatologues sur la sensibilité du climat, ces résultats appellent à une action forte et immédiate pour la transition énergétique. En outre, nous montrons qu'un ralentissement démographique, elle seule, n'évite pas la catastrophe mais ne ferait que repousser l'effondrement.

Pour rappel, ce chapitre met en relation le marché de l'énergie à travers du prix du carbone, et un secteur financier et monétaire par l'endettement du secteur privé non financier. Dans une certaine mesure, le signal prix du carbone va en tout état de cause impacter positivement le prix du pétrole. Dès lors, en faisant l'analogie entre ces deux prix, un résultat supplémentaire de ce chapitre est que celui-ci met en lumière le canal de l'augmentation de l'endettement – source potentielle d'instabilité financière – à travers les trajectoires de l'augmentation du prix de la ressource énergétique. En effet, afin de compenser les effets de l'augmentation des prix, les entreprises vont recourir à l'endettement afin de conserver leur croissance.

1.6 Les prolongements de la thèse

Les trois derniers chapitres qui résument les nouveaux fondements pour la dynamique macroéconomique, est annonciateur de travaux de plus grande ampleur; il conduit vers un programme de travail permettant l'étude de modèles à plus grande échelle. Plusieurs niveaux de prolongements sont possibles, notamment : (i) dans le cadre de la question des canaux de transmission entre les prix de l'énergie et le secteur financier ; (ii) dans la cadre de la continuité des travaux de Meadows (cf. Meadows et al. (1972) et Meadows and of Rome (1974)).

La philosophie initiale de la thèse était de répliquer un environnement économique à travers une modélisation afin d'analyser l'influence des prix de l'énergie dans des économies très financiarisées. Cette observation ne pouvant pas être traitée de manière fine avec les outils dont la thèse disposait initialement, la thèse s'est alors axée vers la création d'outils permettant de construire un tel environnement macrodynamique. Les trois derniers chapitres de la thèse forment une base solide pour le développement d'un cadre d'analyse permettant au cadrage initial donné par le questionnement à la genèse de la thèse, néanmoins, plusieurs travaux restent à faire : (i) le cadre de l'économie ouverte analogue à ce qui a été développé dans le premier papier DSGE où le pétrole, utilisé comme intrant dans la fonction de production, est importé et que son prix est alors exogène ; (ii) l'analyse d'une fonction de production CES avec l'énergie comme intrant additionnel de production ; (iii) une approche multisectorielle (la motivation de cette dernière approche est expliquée ci-dessous).

Le second niveau de prolongements est de plus grande ampleur. Il fait suite aux travaux de Meadows et du Club de Rome en 1972 et 1974. En effet, le dernier article de la thèse reprend les travaux de Nordhaus et du modèle de DICE qui cherche à étudier les impacts de l'activité humaine contemporaine – économie post-industrielle – dans les générations à venir et conclut que l'activité économique peut vraisemblablement se retourner drastiquement avant la fin du siècle. Un article de 2012, publié dans la revue Nature (cf. Barnosky et al. (2012)) lance une alerte à la communauté internationale : les dégradations que le mode de vie de l'humanité inflige aux écosystèmes planétaires provoquent des franchissements de seuil en partie irréversibles et susceptibles de mener à une catastrophe humanitaire. Dans la modélisation de l'activité économique, ces conclusions ne sont pas récentes : dans les années 1970, le rapport *Halte à la croissance* montrait déjà des trajectoires qui nous mènent vers l'effondrement de l'économie dès 2030. Depuis lors, les trajectoires ont été testées par Turner (2008), Turner (2012), Turner (2014) et ont montré une étonnante similarité avec les trajectoires des données statistiques historiques qu'aucun modèle de l'époque n'a été capable de reproduire. Or, le modèle du couple Meadows est très frustre dans son cœur économique. Ce constat amène à conclure qu'il faut réconcilier les modèles macroéconomiques avec les modèles des physiciens. Ceci doit encourager l'économiste-modélisateur à casser la convention qui postule que la conversion d'une quantité physique en un prix n'induit pas de distorsion dans sa valeur économique (cf. les éléments sur le coût des facteurs de production *supra* ou bien Giraud and Kahraman (2015)). Car cette hypothèse amène à des modélisations qui confondent les voitures avec l'usine qui les a produit. Il est donc nécessaire, dans la poursuite des travaux engagés dans la thèse, de mettre en place une analyse multi-sectorielle dans la conceptualisation du système productif de l'économie. Cette vision de l'économie permettrait d'isoler notamment l'énergie et la matière afin de construite des modèles macroéconomiques cohérents avec les réalités physiques (cf. Motesharrei et al. (2014)). Tout ceci contribuerait au développement d'outils macroéconomiques et financiers – avec de la monnaie et de la dette – fournissant des trajectoires de transition énergétique plus en phase avec les contraintes physiques. Dans un aspect plus théorique, des tests de sensibilité des paramètres par le calcul de Malliavin permettraient d'augmenter la crédibilité des résultats proposés par ce programme de recherche.

Ces travaux devront permettre de contribuer aux débats actuels qui sont sans doute les enjeux de notre génération sur : (i) la stagnation séculaire ; (ii) la transition énergétique ; et (iii) l'évaluation de politiques publiques sur le climat.

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Chapter 2

The Effects of Oil Price Shocks in a New-Keynesian Framework with Capital Accumulation

Abstract: The economic implications of oil price shocks have been extensively studied since the 1970s'. Despite this huge literature, no dynamic stochastic general equilibrium model was available that captures two well-known stylized facts: 1) the stagflationary impact of an oil price shock, together with 2) the influence of the energy efficiency of capital on the depth and length of this impact. We build, estimate and simulate a New-Keynesian model with capital accumulation, which takes the case of an economy where oil is imported from abroad, and where these stylized facts can be accounted for. Moreover, the Bayesian estimation of the model on the U.S. economy (1984-2007) suggests that the output elasticity of oil might have been above 10%, stressing the role of oil use in U.S. growth at this time. Finally, our simulations confirm that an increase in energy efficiency significantly attenuates the effects of an oil shock—a possible explanation of why the third oil shock (1999-2008) did not have the same macro-economic impact as the first two ones. These results suggest that oil consumption and energy efficiency have been two major engines for U.S. growth in the last three decades.

JEL Codes:: C68, E12, E23, Q43

Keywords: New-Keynesian model, DSGE, oil, capital accumulation, stagflation, energy efficiency.

2.1 Introduction

The two episodes of low growth, high unemployment, low real wages and high inflation that characterized most industrialized economies in the mid and late 1970s' are usually viewed as the paradigmatic consequences of large price "shocks" that affect various countries simultaneously.[1] Despite the huge literature devoted to the implications of oil prices, to the best of our knowledge, no dynamic general equilibrium model was available that captures the next two stylized facts: 1) the stagflationary impact of sharp oil real price rise, together with 2) the various impacts of capital accumulation: in addition to the well-known hysteresis effect (Khramov, 2012), the potential role of capital as a new channel for monetary policy through the non-arbitrage relation involving the rental rate of capital and the Central Bank's interest rate and, above all, the role of capital energy efficiency in dampening the impact of an oil price rise.

The present paper introduces energy into an otherwise standard New Keynesian model in the same way as Blanchard and Galí (2009) and Blanchard and Riggi (2013), to which it adds capital accumulation. Energy is understood as being just oil, which is imported from abroad at an exogenous world price. Oil imports are paid for with exports of output. For simplicity, the balance of trade is assumed balanced at every date, so that exports adjust to the cost of imports.[2] Oil is consumed by households and used as an input in the production of intermediate goods. As a matter of fact, and this might be viewed as the main contribution of this paper, when estimated on the U.S. (1984-2007), the output elasticity of oil use turns out to be significantly larger than what is currently assumed in the macro-economic literature.[3] More specifically, we find an elasticity between 11% and 12%. In particular, this is much higher than the cost share of oil, which is usually less than 3%. Our finding confirms the standpoint that has been defended by several authors, including Kümmel et al. (2010), Kümmel (2011) and Ayres

[1] As was rightly noticed by one anonymous referee, the word "shock" is misleading despite its widespread use. The first oil "shock" was considered by some as a possibility as early as December, 1971, and by May, 1973, had become the single most likely scenario focussed upon by one major oil MNC. The second oil "shock" was first considered as a possibility in March, 1976, and in September, 1976, was vigorously discussed in a scenario group meeting under the heading 'Producer Miscalculation/Middle East "accident'. The accident focussed by some was the downfall of the Shah. The 1999/2008 "shock" really took off post-9/11. That said, given the conventional methodology of DGSE models, where exogenous events, such as sharp oil price rise, are treated as shocks, we have opted to bear with this terminology.

[2] As in Blanchard and Galí (2009) and Blanchard and Riggi (2013), we assume that the real price of imports (oil) is some exogenous stochastic process. The exchange rate is therefore not explicitly modeled.

[3] One can note that between 1984 and 2007 the U.S. economy was mostly an oil importer country. However, the assumption that U.S. only use imported oil does not interfere with the estimation of the output elasticity of oil. Indeed, the latter involves only GDP growth and the growth of oil usage. Whether the oil used is domestically produced or imported from abroad is independent from our elasticity estimation.

and Voudouris (2014), according to whom the importance of energy in the fabric of economic growth is amply underestimated in the traditional Solowian approach.[4]

As a result, our specification does react to an oil shock by a short-run decrease in real GDP and some inflation.[5] Next, the introduction of capital accumulation turns out not to impair the stylized facts just alluded to. Capital even amplifies the response of the economy to an oil real price rise. Our third, and most important, conclusion is that a reduction of output elasticity of energy suffices to imply a significant reduction of the effect of a shock on macroeconomic performances. This is the way the reduction of the sensitivity of industrialized countries to the oil price rise in the 2000s' is accounted for in this paper.

When addressing these issues, we keep an eye on the events of the past decade that seem to call into question the relevance of oil price changes as a significant source of economic fluctuations. Since the late 1990s' indeed, the global economy has experienced an oil shock of sign and magnitude comparable to those of the 1970s' but, in contrast with the latter episodes, GDP growth and inflation have remained relatively stable in much of the industrialized world until the financial turbulences of 2007-2009 (cf. e.g., Sánchez (2008), Blanchard and Galí (2009), Kilian (2008), Hamilton (2009)). In Blanchard and Galí (2009), a structural VAR analysis suggests that the effects of oil price rises have recently weakened because of the decrease in real wage rigidities, a smaller oil share in production and consumption, and improvements in the credibility of monetary policy. While these three properties did most probably play a role, this paper explores the explanatory power of yet another channel —namely the change in energy efficiency in the industrial sector during the 1980s', as a consequence of the first two oil shocks. At first glance, it seems that the impact of energy efficiency is already taken into account through the decline of energy share in added value, analyzed in Blanchard and Galí (2009).[6] As we argue in the next section, however, these two parameters —cost share and energy elasticity— should be viewed as decoupled variables, in general. Consequently, if the energy efficiency of a country can no more be captured through its energy cost share, we need an explicit modeling of the efficiency of capital. This is yet another motivation for having added a capital accumulation dynamics to the standard DSGE model.[7] The

[4]Thus, the addition of capital is important not just because it adds realism to the modeling approach, but also because it improves the reliability of our empirical estimation of output elasticity with respect to energy. Absent capital, this elasticity could be suspected to capture the (hidden) indirect spillover of capital.

[5]For simplicity, we did not add a growth trend to our model. Were we to do so, our results would be a short-run decrease of the real GDP with respect to the long-run growth trend.

[6]Profit-maximization and perfect competition in frictionless markets imply indeed the equality of energy output elasticity with the cost share of energy. Since the inverse of output elasticity may be taken as a proxy for energy efficiency, it might seem that the improvement of energy efficiency is reflected through the decline of energy share.

[7]See, e.g., Khramov (2012) and the references therein.

decoupling of energy efficiency from the energy share cost then opens the door for a reexamination of why the 2000s' have been so different from the 1970s'.

Our findings are twofold: 1) the improvement of energy efficiency might well have been a powerful explanatory factor for the muted impact of the rise in real oil price experienced during the early 2000's in comparison with the 1970s', but 2) oil elasticity did not decrease during the first decade of this century in the U.S. We make the first point by studying the impulse response function of our DSGE model. And the second is made by various estimations of oil output elasticity within different time intervals. As a consequence, one reason for the muted impact of the third oil price sharp increase can indeed be attributed to the significant improvement in oil use efficiency, which, as we show, occurred in the U.S. around 1979. But such a progress did not take place later on.[8]

Among the 2000s' oil shock literature, recently, in addition to Blanchard and Galí (2009), a last contribution is worth noticing, namely Blanchard and Riggi (2013). The latter performs an estimation of a Macroeconomic DSGE model, and confirms that a large decrease of real wage rigidities and an increase of the credibility of the monetary policy must have contributed to dampening the shock. Together with an estimation based on indirect inference, Blanchard and Riggi (2013) *calibrate* the production function as being constant return to scale with an output elasticity of oil set equal to 0.015 for the period pre-1984 and 0.012 for the post-1984 period, the output elasticity of labor being therefore 0.985 and 0.988, respectively.

By contrast, in the present paper, using a Bayesian approach, most parameters are estimated, including oil's output elasticity. The latter turns out to lie between 0.11 and 0.12. That is, a 10% increase of oil consumption leads to a 1.1% or 1.2% increase of output —an elasticity 10 times larger than the one supposed by Blanchard and Riggi (2013). Our finding also contrasts with the literature where oil output elasticity is usually identified with the energy cost share, hence close to 0.03. Where does the gap between our output elasticity, α_e, and the cost share come from? In the present DSGE model, it arises from the GDP definition and Calvo viscosity of prices (whose value, as usual, is calibrated around 0.65) which prevents prices from reflecting the standard first order conditions from which the cost share theorem is derived. At the stationary state, the Calvo friction vanishes, and one has:

[8] This is in line with one of the conclusions in Blanchard and Galí (2009) and Blanchard and Riggi (2013), where, based on the postulated identity between the cost share and oil elasticity, a presumed reduction of the latter was identified as an explanatory candidate. The difference is that we identify the late 1970s' as being the unique moment where a significant break through in oil efficiency took place, while these authors conclude from the decline of the oil cost share in the early 2000s', that oil efficiency also recently improved.

$$\begin{aligned}\text{Oil's cost share} :=& \frac{P_e E}{P_y Y} \\ =& \frac{P_e E}{P_q Q - P_e E} \\ =& \frac{\alpha_e}{\mathcal{M}_p - \alpha_e}\end{aligned} \tag{2.1}$$

where Y stands for GDP, Q for domestic output, E for oil and $\mathcal{M}_p$ for the price markup in the (imperfectly competitive) production sector. So that, even though they never coincide, the cost share and output elasticity remain somewhat close to each other. But along the transitional dynamics *towards* the steady state, the Calvo friction *does* enters in the scene. And this transitional dynamics is crucial for the Bayesian estimation of the output elasticity of energy. Together with the fact that, contrary to a large body of the literature, we do not restrict returns to scale to be *a priori* constant, this explains why our Bayesian estimation does not lead to an elasticity close to the empirically observed cost share. Conversely, (2.1) implies that, along the steady state, absent any price friction, the cost share should be close to 10%, which is obviously at odds with historical data. This simply confirms that the U.S. economy evolved rather far from its steady state during the period under scrutiny. The value of elasticity parameter, α_e, being constant, its value does not depend upon whether the economy remained in the vicinity of its steady-state path, or not.

The paper is organized as follows. The next section presents the conceptual framework, in particular the decoupling issue just alluded to. Section 3 describes the model. Section 4 provides the estimation procedure. Section 5 gives our main findings by analyzing at length how our model reacts to a real oil price rise. We leave the complete methodological details and numerical simulations to an extensive on-line Appendix.

2.2 Conceptual Framework

Apart from the introduction of energy as an input in the aggregate production function, our New-Keynesian framework is rather standard. Three conceptual issues are worth being addressed: the possible decoupling between the cost share and output elasticity of energy (subsection 2.1.), the introduction of increasing returns to scale (section 2.2.) and the very definition of a global "price level" (subsection 2.3.)

2.2.1 Decoupling the Cost Share from Output Elasticity

Following, e.g., Kümmel et al. (2010), let us mention yet another reason why output elasticity might not always equal the profit share, even in competitive markets, under constant returns to scale and absent any externality of omitted variables. Denoting $x = (x_i)_i$ the input vector, $Y(\cdot)$ the production function, and $p = (p_i)_i$ the real price of inputs, the profit maximization program of the producer,[9]

$$\max_x \; Y(x) - p \cdot x \tag{2.2}$$

leads to:

$$\varepsilon_i := \frac{x_i}{Y(x)} \times \frac{\partial Y}{\partial x_i}(x) = \frac{p_i x_i}{p \cdot x} \tag{2.3}$$

where ε_i is the output elasticity of the production factor, x_i. This textbook argument rests on the assumption that the producer's maximization program (2.2) faces *no* constraint apart from the very definition of $Y(\cdot)$. Suppose, on the contrary, that (2.2) must be written, somewhat more realistically:

$$\max_x \; Y(x) - p \cdot x \quad \text{ s.t. } \; f(x) = 0 \tag{2.4}$$

where $f(\cdot)$ is some smooth function. Whenever the input, x_i, is interpreted as energy, we can think of $f(\cdot)$ as capturing geological resource restrictions on fossil energies, geopolitical or climatic constraints, the bargaining power of labor forces, institutional rigidities of the labor market, etc. The cost-share identity (2.3) now involves a shadow price given by the (normalized) Lagrange multiplier, λ, of the additional constraint, $f(x) = 0$:

$$\varepsilon_i = \frac{x_i\big(p_i - \lambda \frac{\partial f(x)}{\partial x_i}\big)}{p \cdot x - \lambda x_i \frac{\partial f(x)}{\partial x_i}}. \tag{2.5}$$

It follows that shadow prices may be responsible for the decoupling between the energy share, $p_i x_i / p \cdot x$, and its output elasticity, ε. Suppose, for instance, that the cost share remains small, while $\lambda \to +\infty$. Then, $\varepsilon_i \to 1$.[10] Similarly, ε may take any real value between $x_i p_i / x \cdot p$ and $-\infty$ whenever $0 < \lambda < (p \cdot x) \frac{\partial x_i}{\partial f(x)}$. So that a large share $x_i p_i / x \cdot p$ is compatible with a small ε. The strength of this latter argument for decoupling is that

[9] In the following argument, the output is taken as numéraire.

[10] A similar observation is made in Kümmel et al. (2008) and Kümmel (2011).

it prevents us from concluding that one factor's return is underpaid (when the profit share is below its output elasticity) or overpaid (in the opposite situation): both might well exhibit a "fair return" once all the constraints in the production sector have been taken into account.[11]

In a companion paper, Kahraman and Giraud (2014), the elasticity of primary energy use is estimated through an error correction model for 33 countries, along time series from 1970 to 2011. Estimated elasticities are robustly located between 0.4 (France) and 0.7 (U.S.), with an average around 0.6 —which means that the mere identification of the cost share with output elasticity induces an underestimation of the latter by a factor 6 to 8. Although the method for reaching it is entirely different, this empirical finding is in accordance with the Bayesian estimation of the present model —where, the output elasticity of oil turns out to be close to 0.12, implying an undervaluation by a factor 4 to 6. According to the previous argument, this seems to suggest that economic actors face binding constraints regarding the use of primary energy. Similarly, in Kahraman and Giraud (2014), it is suggested that the output elasticity of capital could be much lower than is suggested by the capital share. As already noticed, this is perfectly compatible with (2.5).

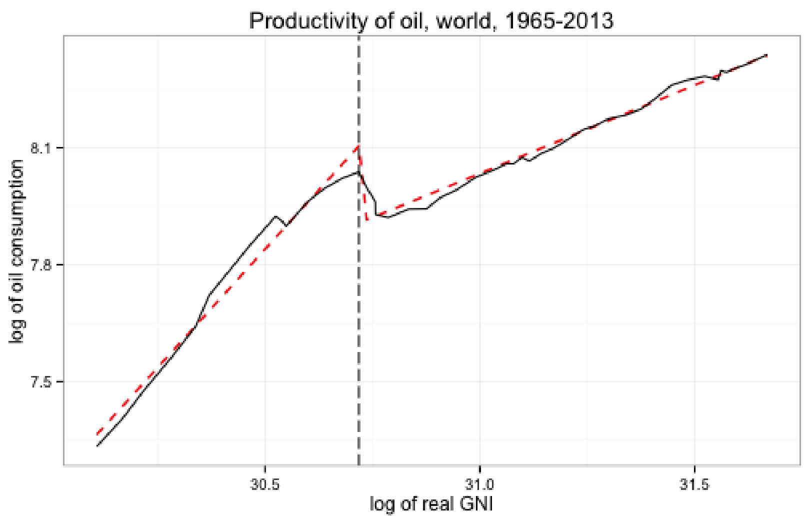

Figure 2.1: Productivity of Oil, World, 1965-2013

Sources: British-Petroleum-Company (2013) for oil series and World-Bank (2013) for World Real GNI

[11]One may be tempted to replicate that, at least in the perfectly flexible case, prices should already reflect the constraint, $f(x) = 0$, so that there would be no need to make it explicitly in (2.4). However, for prices to convey publicly this information, some individual producers must hold it privately, that is, they must have taken it into account in their individual profit-maximization programme. Consequently, it must show up as well at the aggregate level.

Put otherwise, one purpose of this paper is to confirm the relevance of the reduction of the dependence of the industrialized economies (especially the U.S. economy) to oil in the 2000s' as compared to the 1970s' as an explanatory factor for why the last decade was so different from the 1970s', and to show that this can be illustrated and modeled without relying on any *a priori* coupling between the output elasticity and cost share.[12]

As shown by Figure 2.1 this dependency significantly decreased during the 1980s', most probably as a consequence of the adaptation of the industrial sector to the shocks of the previous decade. The figure plots with a yearly frequency the values of oil world consumption in millions of tones on the x-axis against world GNI in 2005 constant U.S. dollars on the y-axis. The first point on the south-west stands for 1965, the last one (in the north-east), for 2013. The dashed vertical black line corresponds to the structural break found in 1979.[13] The dashed red line represents fitted values of the regression. A complete independence between the world GNI and oil consumption at the world level would imply a vertical segment. Clearly, there was an improvement in the energy efficiency from 1979 as compared to the 1960's and 1970s': the slope (1.215) of the affine segment prior to the second oil price increase is more than twice as large as the posterior 1979 counterpart (0.4534).[14] In this paper, we capture such a shift by decreasing the output elasticity of oil.[15] The main question addressed in this paper is whether such a change in the structure of the production sector can be responsible for the muted impact of the third oil price rise on Western economies.

2.2.2 Increasing Returns?

On the analytical side, the main consequence of allowing for such non-conventional elasticities is to force us to relax the textbook constant returns assumption which underlies (2.3). There are various motivations for not imposing *a priori* the constant returns to scale restriction. First, as is well known, the fact that empirical investigations often conclude that, at the aggregate level, returns to scale seem constant is but an econometric artifact.[16] Second, it is equally well-known that a production sector with strictly decreasing returns to scale cannot exhibit indefinite growth.[17] Hence, endogenous growth

[12]Thus, this work complements Blanchard and Galí (2009) whose analysis is based on this coupling.

[13]The optimal break point has been found following Bai and Perron (2003): optimal segments are identified by minimizing the Residual Sum of Squares and the Bayesian Information Criterion. The OLS estimation is given by adding dummies variables at the break date.

[14]Of course, the large $adjusted - R^2$ (between 0.9762 and 0.9768) obtained in the two OLS performed before and after 1979 suggest a strong endogeneity between GNI growth and oil consumption. This issue is addressed in Kahraman and Giraud (2014).

[15]Indeed, whenever $\partial Y(x)/x$ is a decreasing function of x (as in our model), an increase of the productivity, Q/x, of some input x together with an increase of the demand for x, translates into a decrease of ε according to (2.5).

[16]Cf. Samuelson (1979).

[17]Cf. Hurwicz and Reiter (1973).

must rely, at some stage or another, on some non-convexity in the production sector. Third, empirical inquiries unambiguously conclude that, a the micro level, most industrial sectors exhibit increasing returns to scale.[18]

Our purpose is not to revisit here the pros and cons of exogenous versus endogenous growth theory. More simply, we aim at letting the data speak by estimating a New Keynesian model that is compatible with every kinds of returns to scale. Of course, when dealing with a non-concave production function, the main challenge is to define the producer's behavior, as the mere profit maximization program (2.2) or (2.4) may no more have any solution (or may admit several solutions). We adopt the most commonly used behavioral framework found in the literature devoted to increasing returns, namely marginal cost pricing:[19] At equilibrium, the representative producer chooses a production plan, x, such that the price, p, of inputs, equals their marginal cost, $\partial Y(x)/\partial x_i$. This seems to be the less onerous way of dealing with the lack of decreasing returns. Indeed, marginal cost pricing readily leads to the familiar first-order conditions that are otherwise instrumental for the numerical simulations of the response to exogenous shocks in the DSGE tradition. In the context of the DSGE literature, our departure with the standard practice is therefore that first-order conditions are necessary but need no more be sufficient for profit-maximization.

As a matter of fact, our estimation concludes unambiguously that returns to scale are strictly increasing.

2.2.3 GDP Deflator and CPI

With no capital accumulation and zero public expenditures, our model reduces to the one first introduced by Blanchard and Galí (2009) with two changes, in the monetary policy and in the definition of the GDP deflator. In Blanchard and Galí (2009), indeed, the CPI is defined as $P_{c,t}$, the core CPI, as $P_{q,t}$ and the GDP deflator, as $P_{y,t}$. In that paper, the three indices are related by the following equations:

$$P_{q,t} := P_{y,t}^{1-\alpha_e} P_{e,t}^{\alpha_e} \tag{2.6}$$

$$P_{c,t} := P_{q,t}^{1-x} P_{e,t}^{x} \tag{2.7}$$

and as a consequence of (2.6):

$$P_{y,t} = P_{q,t}^{\beta} P_{e,t}^{1-\beta} \tag{2.8}$$

[18] Blinder et al. (1998) and Eiteman and Guthrie (1952).

[19] Cf. Cornet (1988), Quinzii (1992), Beato and Mas-Colell (1985), Bonnisseau and Cornet (1990), Dehez and Dreze (1988) or Giraud (2001).

where $P_{e,t}/P_{q,t}$ is the (exogenous) real price of energy at time t, $\alpha_e, x \in (0,1)$, but — and this turns out to be crucial—, $\beta > 1$.

These conventions have the paradoxical consequence that, when the energy price experiences an upward shock, the GDP deflator decreases (everything else being kept fixed) as can be seen from (2.8). We fix this problem by imposing $P_{c,t} \equiv P_{y,t}$ while keeping (2.7), and the following budget identity, which defines GDP, in the left-hand side, as the aggregation of domestic product minus energy import:

$$P_{y,t}Y_t = P_{q,t}Q_t - P_{e,t}E_t.$$

2.3 A New-Keynesian Economy with Imported Energy

Let us denote $P_{k,t}$ and $P_{e,t}$ the nominal price of capital and oil respectively. We define real prices relative to the price of final by:

$$\begin{aligned} S_{e,t} :=& \frac{P_{e,t}}{P_{q,t}} \\ S_{k,t} :=& \frac{P_{k,t}}{P_{q,t}}. \end{aligned}$$

They both are assumed to follow AR(1) processes:

$$\begin{aligned} \ln(S_{e,t}) =& (1-\rho_{se})ln(S_e) + \rho_{se}\ln(S_{e,t-1}) + e_{se,t} \\ \ln(S_{k,t}) =& (1-\rho_{sk})ln(S_k) + \rho_{sk}\ln(S_{k,t-1}) + e_{sk,t}. \end{aligned}$$

where $e_{se,t} \sim \mathcal{N}(0, \sigma_{se}^2)$ and $e_{sk,t} \sim \mathcal{N}(0, \sigma_{sk}^2)$, S_e and S_k respectively stand for the steady state real price of oil and capital.

Let us now briefly describe how capital accumulation and imported oil enter into the model.

2.3.1 Household

The representative infinitely-lived household works, invests in government bonds (B_t) and capital, pays taxes and consumes both oil and the final good. Its instantaneous utility function is:

$$U(C_t, L_t) = \ln(C_t) - \frac{L_t^{1+\phi}}{1+\phi},$$

where C_t is the consumption at time t, L_t is labor and ϕ is the inverse of the Frisch elasticity. Let W_t denote the nominal wage, $P_{k,t}$, the nominal price of capital, and r_{t+1}^k, the real rental rate of capital. The dynamics of capital accumulation follows, as usually,

$$I_t := K_{t+1} - (1-\delta)K_t,$$

where $\delta \in (0,1)$ is the depreciation rate. At variance with several DSGE models, the capital price is not identified with the consumption price but is rather viewed as exogenous. Indeed, the custom to identify both consumption and capital prices arises, as is well-known from the Cambridge controversy, from the lack of an equilibrium condition that would permit pining down the market value of capital.[20] But this mere identification prevents from capturing decoupled bubble phenomena, such as the housing bubble that affects most Western countries since the middle of the 1990s'.[21]

The nominal short-run interest rate, i_t, is set by the Central Bank. At time t, T_t denotes the tax paid by the household. Being the shareholder of the firms, the household receives the global dividend $D_t := \int_0^1 D_t(j)dj$, i.e., the sum of dividends of all intermediate good firms. Household aims to maximize her lifetime discounted utility function under the following budget constraint:

$$\begin{aligned} P_{e,t}C_{e,t} + P_{q,t}C_{q,t} \quad &+ \quad P_{k,t}(K_{t+1} - (1-\delta)K_t) + B_t \\ &\leq \quad (1+i_{t-1})B_{t-1} + W_tL_t + D_t + r_t^k P_{k,t}K_t + T_t, \end{aligned}$$

where the consumption flow is defined as:

$$C_t := \Theta_x C_{e,t}^x C_{q,t}^{1-x}, \tag{2.9}$$

with $x \in (0,1)$ being the share of oil in consumption and $\Theta_x := x^{-x}(1-x)^{-(1-x)}$. The optimal allocation of expenditures among different domestic goods yields:

$$P_{q,t}C_{q,t} = (1-x)P_{c,t}C_t \tag{2.10}$$

$$P_{e,t}C_{e,t} = xP_{c,t}C_t \tag{2.11}$$

$$\text{where} \quad P_{c,t} = P_{e,t}^x P_{q,t}^{1-x} \quad \text{is the CPI index.} \tag{2.12}$$

[20]Cf. Samuelson (1966).

[21]See, e.g., Bonnet et al. (2014).

The first order conditions of its utility maximization yields:

$$\begin{aligned}
C_t : &\quad U_C(C_t, L_t) = \lambda_t P_{c,t} \\
L_t : &\quad U_L(C_t, L_t) = \lambda_t W_t \\
B_t : &\quad \lambda_t = \beta \mathbb{E}_t \Big[(1+i_t)\lambda_{t+1}\Big] \\
K_{t+1} : &\quad \lambda_t P_{k,t} = \beta \mathbb{E}_t \Big[\lambda_{t+1}\big(r^k_{t+1} + 1 - \delta\big) P_{k,t+1}\Big].
\end{aligned}$$

2.3.2 Final Good Firm

The role of the final good firm is to buy goods from intermediate firms and then repackage them as final goods. These final goods will be resold to households or exported in exchange of oil. There is a continuum, $[0, 1]$, of intermediate goods that serve in producing the consumption commodity. A representative final good producing firm maximizes its profit with no market power. Its CES production function is given by:[22]

$$Q_t = \Big(\int_{[0,1]} Q_t(i)^{\frac{\epsilon-1}{\epsilon}} di \Big)^{\frac{\epsilon}{\epsilon-1}}, \tag{2.13}$$

where $\epsilon > 0$ is the elasticity of substitution among intermediate goods.

The final good firm chooses quantities of intermediate goods, use as input, $(Q_t(i))_{i\in[0,1]}$ in order to maximize its profit.

2.3.3 Intermediate Goods Firms

Each intermediate commodity is produced through a Cobb-Douglas technology involving oil:

$$\begin{aligned}
Q_t(i) &= A_t E_t(i)^{\alpha_e} L_t(i)^{\alpha_\ell} K_t(i)^{\alpha_k} \qquad (2.14)\\
&\quad \alpha_e, \alpha_\ell, \alpha_k \geq 0,
\end{aligned}$$

where A_t is a total productivity factor (TFP) so that its logarithm follows an $AR(1)$ process, $ln(A_t) = \rho_a ln(A_{t-1}) + e_{a,t}$, where $e_{a,t} \sim \mathcal{N}(0, \sigma_a^2)$.

The strategy of firm i can be decomposed in two steps: First, taking prices $P_{e,t}$, $P_{k,t}$, r^k_t, W_t, and demand $Q_t(i)$ as given, firm i chooses quantities of oil $E_t(i)$, labor $L_t(i)$, and capital $K_t(i)$ so as to minimize its cost. Since returns to scale need not be decreasing, $\alpha_e + \alpha_\ell + \alpha_k$ will possibly be larger than 1. As a consequence, in this paper, the producer

[22] For simplicity, no oil is needed to produce the final commodity out of the intermediate goods.

is assumed to follow the marginal cost pricing behavior, which is characterized by the (standard) first-order conditions:

$$\text{marginal cost} = mc_t(i) \quad := \quad \frac{W_t}{\alpha_\ell \dfrac{Q_t(i)}{L_t(i)}} = \frac{r_t^k P_{k,t}}{\alpha_k \dfrac{Q_t(i)}{K_t(i)}} = \frac{P_{e,t}}{\alpha_e \dfrac{Q_t(i)}{E_t(i)}}. \tag{2.15}$$

In order to keep compact notations, we denote $F_t := \Big(\frac{A_t \alpha_e^{\alpha_e} \alpha_\ell^{\alpha_\ell} \alpha_k^{\alpha_k}}{P_{e,t}^{\alpha_e} W_t^{\alpha_\ell} (r_t^k P_{k,t})^{\alpha_k}} \Big)^{\frac{-1}{\alpha_e+\alpha_\ell+\alpha_k}}$, so that

$$\text{cost function:} \qquad cost(Q_t(i)) = (\alpha_e + \alpha_\ell + \alpha_k) F_t Q_t(i)^{\frac{1}{\alpha_e+\alpha_\ell+\alpha_k}}, \tag{2.16}$$

$$\text{marginal cost:} \qquad mc_t(i) = F_t Q_t(i)^{\frac{1}{\alpha_e+\alpha_\ell+\alpha_k}-1}. \tag{2.17}$$

In the second step, each firm sets the price, $P_{q,t}(i)$, so as to maximize its net profit. We assume that prices are set à la Calvo. A fraction, θ, of intermediate good firms cannot reset their prices at time t:

$$P_{q,t}(i) = P_{q,t-1}(i).$$

and a fraction, $1-\theta$, sets its prices optimally:

$$P_{q,t}(i) = P_{q,t}^o(i).$$

Clearly, $P_{q,t}^o(i)$ does not depend upon i, and we can write $P_{q,t}^o(i) = P_{q,t}^o$, for every i. Therefore we have the following "Aggregate Price Relationship:"

$$P_{q,t} = \Big(\theta P_{q,t-1}^{1-\epsilon} + (1-\theta)(P_{q,t}^o)^{1-\epsilon} \Big)^{\frac{1}{1-\epsilon}}. \tag{2.18}$$

At date t, denote $Q_{t,t+k}(i)$ the output at date $t+k$ for firm i that last resets its price in period t. Firm i's problem is:

$$\max_{P_{q,t}(i)} \mathbb{E}_t \Big[\sum_{k=0}^{\infty} \theta^k d_{t,t+k} \big[P_{q,t}(i) Q_{t,t+k}(i) - cost(Q_{t,t+k}(i)) \big] \Big] \tag{2.19}$$

$$\text{subject to} \qquad Q_{t,t+k}(i) = \Big(\frac{P_{q,t}(i)}{P_{q,t+k}} \Big)^{-\epsilon} Q_{t+k}, \quad \forall k \geq 0.$$

Again, this problem does not depend on i, hence $P_{q,t}(i) = P_{q,t}^o$. From the first order condition for $P_{q,t}^o$ we have:

$$\mathbb{E}_t \sum_{k=0}^{\infty} \theta^k d_{t,t+k} Q_{t,t+k}^o \Big[P_{q,t}^o - \mathcal{M}_p mc_{t,t+k}^o \Big] = 0, \tag{2.20}$$

where: $mc^o_{t,t+k} := F_{t+k}(Q^o_{t,t+k})^{\frac{1}{\alpha_e+\alpha_\ell+\alpha_k}-1}$, $Q^o_{t,t+k} = \left(\frac{P^o_{q,t}}{P_{q,t+k}}\right)^{-\epsilon} Q_{t+k}$ for every $k \geq 0$, $\mathcal{M}^p := \frac{\epsilon}{\epsilon-1}$ is the price markup, and $d_{t,t+k}$ is the stochastic discount factor from date t to date $t+k$.

2.3.4 Monetary Policy

Let $\Pi_{q,t}$ be the core inflation. As is usual, the Central Bank sets the nominal short-term interest rate according to the following monetary policy:

$$\frac{1+i_t}{1+\bar{i}} = \left(\frac{\Pi_{q,t}}{\bar{\Pi}}\right)^{\phi_\pi} \left(\frac{Y_t}{\overline{Y}}\right)^{\phi_y} \varepsilon_{i,t}, \tag{2.21}$$

where $\overline{Y}$, $\bar{i}$ and $\bar{\Pi}$ respectively represent the steady state of Y_t, i_t and $\Pi_{q,t}$. The monetary policy stochastic part is: $ln(\varepsilon_{i,t}) = \rho_i ln(\varepsilon_{i,t-1}) + e_{i,t}$, where $e_{i,t} \sim \mathcal{N}(0, {\sigma_i}^2)$.

2.3.5 Government

The Government budget constraint is:

$$(1+i_{t-1})B_{t-1} + G_t = B_t + T_t, \tag{2.22}$$

where G_t is the nominal government spending. We assume that the real government spending $G_{r,t} = \frac{G_t}{P_{q,t}}$ is an exogenous process given by:

$$\ln(G_{r,t}) = (1-\rho_g)\ln(\omega\overline{Q}) + \rho_g \ln(G_{r,t-1}) + \rho_{ag} e_{a,t} + e_{g,t}$$

with ω, the share of output that the government takes for its own spending, $\overline{Q}$ represents the steady state of the domestic output and $e_{g,t} \sim \mathcal{N}(0, \sigma_g^2)$.

At equilibrium, each economic agent solves its maximization problem, all markets clear, and the government budget constraint is fulfilled.

2.3.6 GDP and GDP Deflator

We define real GDP (Y_t) as follows:

$$P_{y,t}Y_t := P_{q,t}Q_t - P_{e,t}E_t$$

where $P_{y,t}$ is the GDP deflator, that we assume to be equal to the CPI, $P_{c,t}$.[23]

2.4 Estimations

2.4.1 Model Estimation

A log-linear version of the model around its steady state is presented in Appendix A. We estimate the model using Bayesian techniques.[24] We choose this estimation technique because, as pointed out by the literature regarding this subject, the Bayesian approach relies on the general equilibrium approach, and outperforms GMM and ML when it comes to small samples.

The data set used for the estimation is composed of six macroeconomic quarterly U.S. variables: the real GDP in chained dollars, the real private fixed investment, the hours worked, inflation, the oil use in production and the Federal Funds rate.[25] The sample goes from 1984:Q1 to 2007:Q1.[26] Due to the model's stationary specification, we detrended the first two series, which are not original stationary, using linear detrending.[27] The remaining series are stationary, so we do not detrend them, but take out their respective mean for the estimation period.

There are 26 parameters, including some which characterize the exogenous shocks. Within the 26 parameters, we fix 5 according to the literature. The discount factor, β, is calibrated at 0.99, while the depreciation rate, δ, is calibrated at 0.025. We set the government spending output share, ω, at 0.18 and we calibrate ϵ at 8. This generates a steady state markup which approximately equals 1.14. Following Blanchard and Galí (2009) we calibrate the share of oil in household consumption, x, at 0.023.

Before estimating, as there is no consensus over the value of the oil output elasticity, α_e, we conduct an identification study of the model defined *supra*. This identification analysis was recently developed by Ratto (2008), Ratto and Pagano (2010), Andrle (2010), Canova and Sala (2009) and Iskrev (2010) among others, and it is implemented in the Dynare's identification toolbox. This methodology is based on sensitivity analysis

[23]See subsection 2.2. for a discussion of this convention.

[24]All estimations are done with Dynare version 4.4.1 Dynare (2011) Two tests are available to check the stability of the sample generation using MCMC algorithm, implemented in Dynare: The MCMC diagnostic (Univariate convergence diagnostic, Brooks and Gelman (1998)) and a comparison between and within moments of multiple chains.

[25]For further explanation about the series sources and transformation, please refer to the Appendix.

[26]The sample time range is motivated by the well-known structural change in 1984 and the beginning of the financial turmoil.

[27]We did not use HP-filter techniques because linear detrending implies more persistent deviation from trend than one-sided HP- filtered data.

of the first two moments used by the model together with the data. It is possible to analyze the perturbation generated by a small change in one of few parameters in relation to the moment.

The identification analysis, available in Appendix A, gives us the following results. First, the higher the oil's output elasticity, the higher the identification strength is. Second, if the parameter α_e is high, the parameter θ looses nearly all its identification strength in relation to the other variables. This explains why we estimate and compare the model with and without estimating θ.

2.4.2 Estimation Results

Most of the priors are borrowed from Smets and Wouters (2007). As for the elasticity parameters, we will use an inverse-gamma distribution. We use this prior for three reasons: first, in order to rely on positive values, second, in order to concentrate the probability mass around the first parameter value, and third, in order to allow an asymmetry in the estimation. The remaining parameters' priors are explained in the Appendix.

We present here just the results obtained in the conditions leading to the best log-marginal density in the two following cases: (a) θ estimated and (b) θ calibrated.[28] The differentiation of these two cases is motivated by three facts: First, as explained previously, parameter θ looses identification strength as soon as we change the starting values for elasticity parameters. Second, the New-Keynesian Philips Curve equation mixes parameters θ, α_e, α_l and α_k. Therefore, in the estimation process, the inference of the Calvo parameter can interfere with the inference of the elasticity parameters. Third, when estimating θ, the posterior mean obtained suggests that $\hat{\theta} \in [0.9320; 0.9751]$.[29] Such an interval suggests a much higher degree of stickiness than is usually found (or assumed) in the literature. Whenever θ is calibrated, we set its value at 0.65, coherently with the literature.

Table 2.1 reports the prior and posterior distributions for each parameter along with the mode, the mean and the 10 and 90 percentiles of the posterior distribution in the two cases just alluded to. These results require several important remarks. First, the value of oil's output elasticity is robustly close to 0.12 in both cases. This contrasts with the oil's output elasticity of 0.015, postulated by Blanchard and Galí (2009). Second, we find evidence for increasing return to scale technology: on average, capital's output elasticity is 0.37, labor's output elasticity is 0.62 and oil's output elasticity is 0.12 —leading to an average sum of 1.1. Third, the two monetary policy coefficients (ϕ_π and ϕ_y) are

[28] The complete results for the 14 estimations performed are available upon request.

[29] Except for one inference, which gives us $\hat{\theta} = 0.5250$.

significantly different in the two cases. On the one hand, when θ is estimated, there is a lower core inflation coefficient (ϕ_π) than that which is originally stated in Taylor (1993), whereas the response to the output gap (ϕ_y) is higher. These results are in line with what is stated in Rudebusch (2006). On the other hand, whenever θ is calibrated, we find that the response to core inflation is close to the estimation in Taylor (1993), while the response coefficient to the output gap is almost irrelevant. This difference is clearly due to the inference of $\theta \approx 0.96$ in the first case, signifying a high rigidity on core CPI price re-setting. Hence, core inflation has a very low probability to reach a high level compared to the calibrated case ($\theta = 0.65$). Since inflation is highly controlled by the Calvo parameter in the first case, the Central Bank has no reason to overreact to inflation. Furthermore, findings on the inverse of Frisch elasticity are truly different. This is not surprising, since we find no consensus over what this value should correspond to in the literature.[30] When θ is estimated, the Frisch elasticity posterior mean equals 1.585 ($\approx 1/0.6308$) and when θ is calibrated it is equal to 0.79 ($\approx 1/1.2625$). It is worth emphasizing that in Smets and Wouters (2007) the posterior mean for this parameter is 0.52 ($\approx 1/1.92$), close to what we find in the case where θ is calibrated.

Table 2.1: Prior and Posterior Distribution of Structural Parameters

Parameter		Prior distribution	Posterior distribution			
			Mode	Mean	10%	90%
θ estimated						
Capital elasticity	α_k	IGamma(0.1,2)	0.3728	0.3599	0.3380	0.3822
Labor elasticity	α_ℓ	IGamma(0.4,2)	0.6424	0.6411	0.6111	0.6745
Oil elasticity	α_e	IGamma(0.6,2)	0.1234	0.1254	0.1051	0.1460
Inverse Frisch elasticity	ϕ	IGamma(1.17,0.5)	0.6209	0.6308	0.4736	0.8019
Taylor rule response to inflation	ϕ_π	Normal(1.2,0.1)	1.2235	1.2253	1.0686	1.3558
Taylor rule response to output	ϕ_y	Normal(0.5,0.1)	0.8020	0.7882	0.6884	0.8876
Calvo price parameter	θ	Beta(0.5,0.1)	0.9812	0.9812	0.9380	0.9883
θ calibrated						
Capital elasticity	α_k	IGamma(0.2,2)	0.3918	0.3809	0.3624	0.3989
Labor elasticity	α_ℓ	IGamma(0.4,2)	0.5947	0.5966	0.5622	0.6305
Oil elasticity	α_e	IGamma(0.5,2)	0.1132	0.1177	0.0915	0.1434
Inverse Frisch elasticity	ϕ	IGamma(1.17,0.5)	1.2562	1.2625	0.9073	1.6069
Taylor rule response to inflation	ϕ_π	Normal(1.2,0.1)	1.5236	1.5307	1.3883	1.6722
Taylor rule response to output	ϕ_y	Normal(0.5,0.1)	0.0265	0.0214	0.0001	0.0402

[30] See Browning et al. (1999) and references therein

2.5 Simulations and Results

2.5.1 The Effects of an Oil Shock

There are six sources of potential exogenous shocks in our economy: real price of oil, real price of capital, government expenditure, monetary policy, price markup and the technology. Having estimated the model, we study the impulse response functions (thereafter IRFs), using the mean of the posterior estimation. We will concentrate on the real price of oil shock.

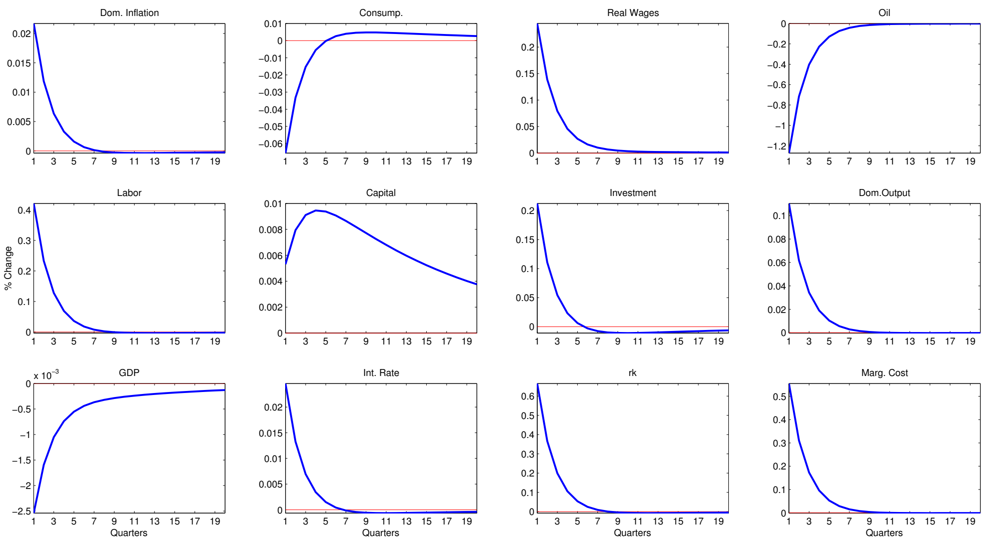

FIGURE 2.2: Response to one Standard Deviation Shock on Real Price of Oil. Case: θ estimated

We make the analysis for both estimation protocols, namely the situation where θ is estimated along with the other parameters, and its counterpart where θ is calibrated.

Let us begin with the case where θ is estimated. The estimated value of θ being 0.96, this implies a high stickiness level in prices. The corresponding IRFs are represented in Figure 2.2. As expected, an increase of the price of oil generates a immediate decrease of oil consumption but a limited reaction in domestic prices (which are too viscous to react instantaneously). Consequently, intermediate firms do not reduce their production, but prefer substitute capital and labor to oil. Real wages, therefore, increase as well as the rental rate of capital. Because domestic consumption is in any case affected and the rental rate of capital is high, the representative household prefers to invest more than to consume. However, despite the increase in domestic production, GDP is affected

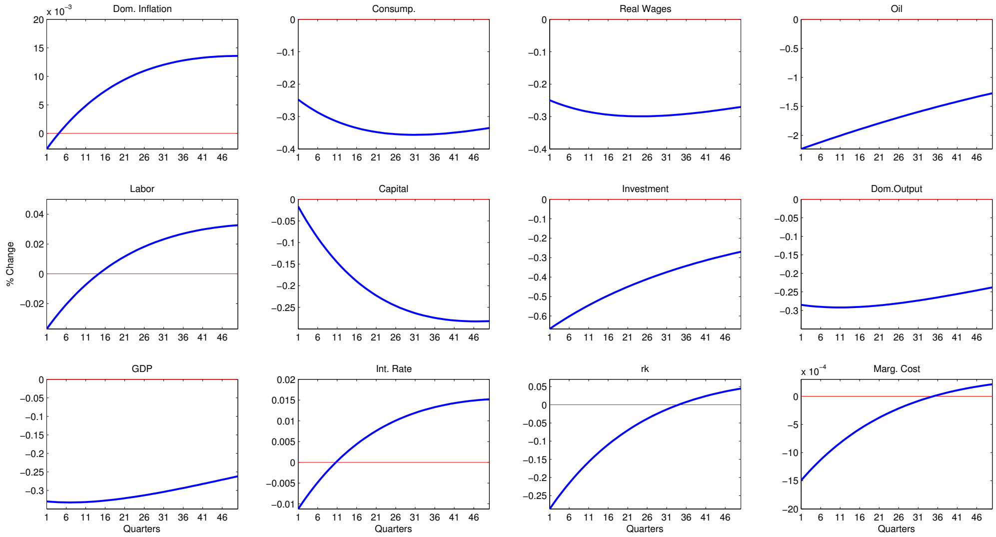

FIGURE 2.3: Response to one Standard Deviation Shock on Real Price of Oil. Case: θ calibrated

negatively because of the growing cost of importing oil. Finally, the small inflationary pressure induces a weak monetary reaction of the Central Bank.

Let us now study the case where θ is calibrated at 0.65. Figure 2.3 presents the corresponding IRFs. Now, a larger fraction of firms can reset optimally their prices instantaneously, so that the inflationary effect of an oil shock is more pronounced. Therefore, the shock provokes a large decrease in consumption. The latter hits the domestic producers, who therefore reduce their production. Due to the lower demand, no substitution effect takes place, so that firms decrease their demand for capital, labor and oil. This reduced inputs' demand depressed both real wages and the rental rate of capital. Consequently investment decreases. The reduction of production lowers also the marginal cost of inputs, which provokes a deflation. In an attempt to re-launch the economy, the Central Bank then decreases its interest rate. GDP is also more negatively affected in this case than in the previous simulations, due to the reduction of domestic output. Moreover, the negative impact of the oil shock is much more persistent in the calibrated case. This comes from the fact that the posterior value of the autoregressive shock drops from 0.9872 to 0.56. Note also that when θ is calibrated, the posterior value of the Taylor rule response to output parameter, ϕ_y, drops from 0.78 to 0.02, meaning that the Central Bank practically does not react to the GDP reduction.

The overall conclusion is that, contrary to what intuition would perhaps suggest, a higher price flexibility does not imply that the economy is more immune to an oil shock.

2.5.2 Reducing the Oil Dependency?

Let us now analyze the reaction to an oil shock of an economy that has reduced its dependence with respect to oil. To capture this feature, we decrease the oil's output elasticity from 0.11 (or 0.12) to 0.05. Figure 2.4 and Figure 2.5 present the IRFs for one standard deviation increase (1.94%) in the real price of oil in the θ-calibrated case and the θ-estimated situation. In both cases, the impact of an oil shock is significantly reduced by the smaller dependency of the economy with respect to oil: domestic inflation, reduction of real wages and of GDP are attenuated by the reduction of α_e. This is quite logical and should not come as a surprise. Although very intuitive, the finding still sheds light on two issues. First, this provides a good explanatory candidate for the muted impact of the third oil shock which lasted from 2000 to 2007. Our empirical estimation of α_e is based on the whole period 1984-2007, and arises therefore as a time average. This does not preclude the true elasticity of oil from having decreased across time during this very period, as it is convincingly suggested by Figure 3 above. Therefore, one reason why we did not observe the stagflationary effect we could have expected during the early 2000s' may have been the successful reduction of the U.S. economy's dependency towards oil consumption. The second insight is forward-looking. As we have seen, more flexibility does not mean a better immunization against an oil shock, at least if flexibility refers to domestic price flexibility. Reducing the output elasticity of oil, by means of increasing the efficiency of the use of oil, turns out to be a much more promising policy recommendation.

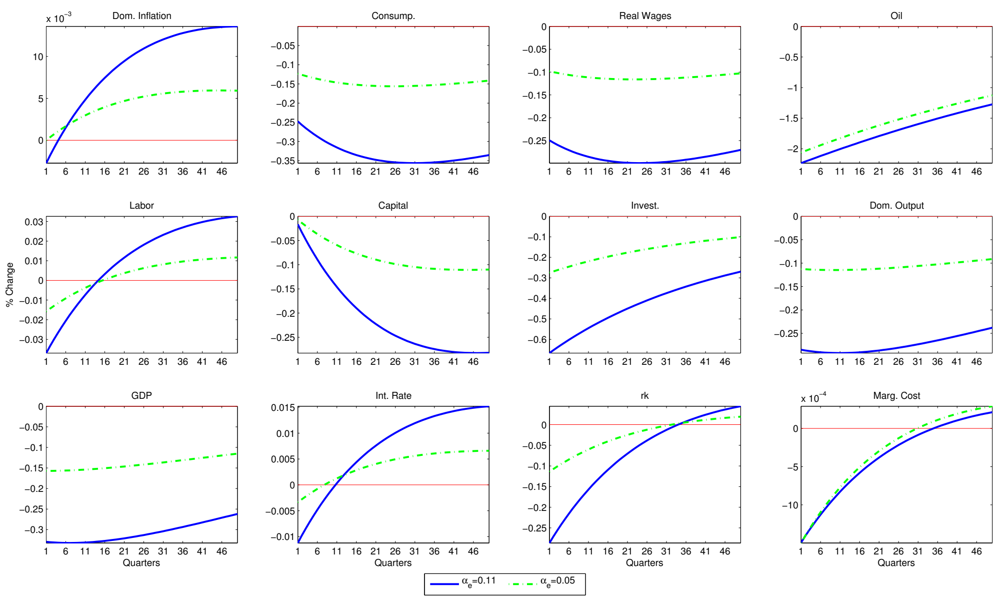

FIGURE 2.4: The Ecological Transition Effect. Case: θ Calibrated

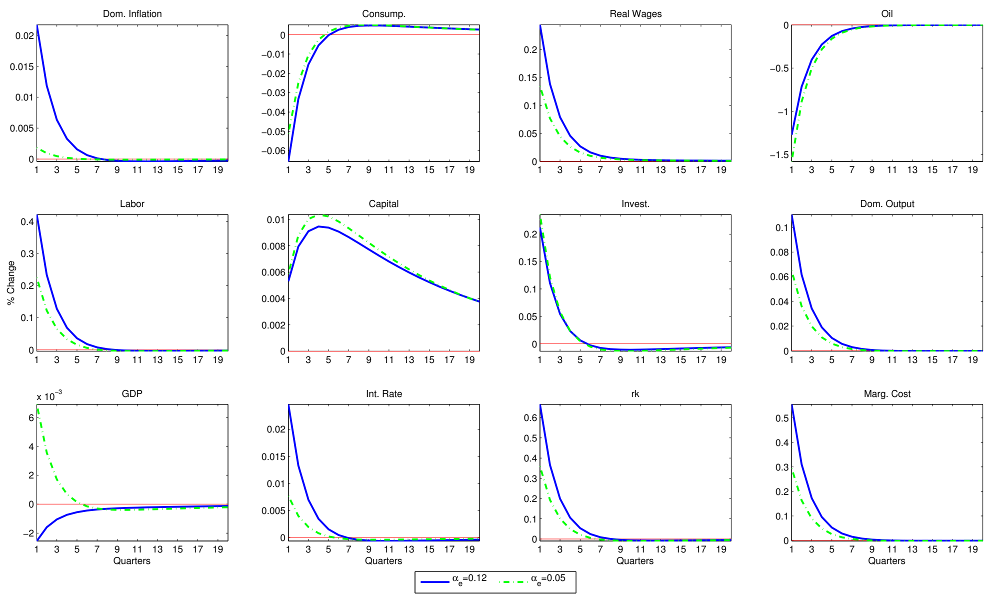

FIGURE 2.5: The Ecological Transition Effect. Case: θ Estimated

2.5.3 How did Output Elasticity of Oil Evolve between 1999 and 2007?

In Section 2.2.1, we argued that there has been an increase in oil productivity during the 1980s'. In Section 2.5.2 we have shown that the reduction in sensitivity of the U.S. economy to the oil shock in the 2000s' could be accounted to a decrease of the output elasticity of oil. A natural question that arises is if oil productivity has continued to increase since then. In order to test this hypothesis, we estimate the model for different time periods, starting from 1984:Q1, by expanding the ending point of the time window (starting from 1999:Q1 to 2006:Q3).

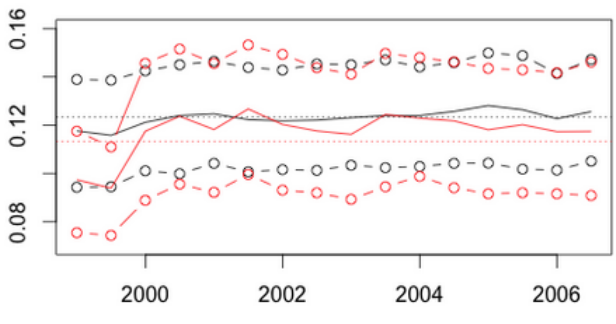

FIGURE 2.6: The Evolution of $\hat{\alpha}_e$ from 1999:Q1 to 2006:Q3 in Bi-annual Frequency.

Figure 2.6 provides the evolution of the estimated $\hat{\alpha}_e$ parameter. The black (resp. red) solid line yields the θ-estimated case (resp. the θ-calibrated case), each point representing the estimation of the model from 1984:Q1 to the date point. For instance, the first value of the black solid line (0.1176) represents the value of α_e from the estimation (including θ) of the model over the period 1984:Q1-1999:Q1, whereas the last point on the red solid line represents the estimated α_e over the period 1984:Q1-2006:Q3 (with θ being calibrated). The black (resp. red) point lines yield the 90% confidence interval and the dotted line is the value of $\hat{\alpha}_e$ estimated over the full sample.

A decrease of oil dependency during 1999-2007 would have resulted in the black (resp. red) solid line lying above the black (resp. red) dotted line. The results suggested by Figure 2.6 however, reflect precisely the opposite phenomenon. To give an example, the first point of the black solid line (i.e., the value of $\hat{\alpha}_e$ estimated over the period 1984-1999) is slightly lower than its value over the period 1984-2007, suggesting that the economy did not reduce its dependency with regard to oil during the period 1999-2007. Adding the sample 1999:Q2-2007:Q1, makes the situation even worse. These findings, of course, require further investigation. In any case, our estimated values and their 90% confidence intervals remain stable over time, meaning that no great shift in efficiency of the use of oil, as the one experienced in 1979, has occurred during that period of time.

2.6 Concluding Remarks

The reasons why the 2000s have been so different from the 1970s —even though both decades experienced a sharp increase in the real oil price, of similar magnitudes — remains an open question. Several assumptions have been suggested in the literature, among which a possible increase in oil efficiency. A proof that the mute impact of the third oil shock in the early 2000s was indeed due, at least partially, and at least in the leading U.S. economy, to an improvement in our usage of oil, would be good news in terms of economic policy. Indeed, both the climate challenge and the possible scarcity of (the daily flow of) fossil fuels in a not too far future imply that one main concern of economic policy, today, is to drive our economies along paths where we can reduce our consumption of oil without impairing economic prosperity. Were we to have been recently successful in putting such a decoupling into practice would therefore suggest that the U.S. economy is on the right way.

Such a claim has been made, indeed, by Blanchard and Galí (2009) and Blanchard and Riggi (2013). It was based, however, on the identification of oil efficiency and the cost share of oil. Following authors like Ayres and Kümmel, we have recalled why such an identification might go astray, calling for a reexamination of the empirical estimation of oil efficiency. The Bayesian estimation of a standard DSGE model has then delivered the following surprising results.

First, the estimation of the output elasticity of oil (as a measure of oil efficiency) turns out to provide much higher estimates than the ones obtained with the conventional computation based on the cost share. The difference appears to be of a factor 6 to 8, in accordance with the findings obtained independently by Kahraman and Giraud (2014). Thus, the U.S. dependency with respect to oil consumption is probably much higher than is usually thought of. Second, our results suggest that aggregate returns to scale are increasing.

Next, we have shown that, even though there has been indeed a significant improvement in oil efficiency around 1979, there is no empirical evidence that this has been the case in the 2000s'. If anything, the output elasticity of oil rather increased during the first decade of this century. If they are confirmed, these results suggest that, apart from the fortunate inheritance of the break through that occurred in the late 1970s', we have to find the reasons for the specificity of the 2000s' elsewhere: possibly in the credibility of monetary policy or the labor market flexibility (as suggested by Blanchard and Galí (2009)) or in the characteristics of a demand shock, as opposed to a supply shock —as suggested by Hamilton (2009) and Kilian (2009). A third line of investigation could as well be explored: What if the low impact of the third oil shock had been due to the huge development of financial markets (in comparison with the 1970s') and, thanks to the Fed's expansionist policy, the easiness with which many actors could compensate for the high price of oil with more debt? This would mean that the real bill of the third oil shock has been paid during the 2007-2009 turmoil.

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Chapter 3

Testing Goodwin with a Stochastic Differential Approach–The United States (1948-2015)

Abstract: This paper follows Harvie (2000)'s research program in testing both Goodwin (1967)'s predator-prey model and the extension proposed by van der Ploeg (1985). The author's aim is to provide a guideline for the estimation and the backtesting strategy that can be applied to such a class of continuous-time macroeconomic model. The goal of this paper is to propose and test stochastic differential equations for Goodwin's model and one of its extension by using an estimation technique based on simulated maximum likelihood developed by Durham and Gallant (2002). The data considered here is that of wage share and employment rate in the United States from 1948:Q1 to 2015:Q4. Results show that models with two structural breaks and endowed with a CES production technology more accurately explains the behavior captured by this data than the Goodwin's Leontief production function. These results are partially confirmed by a backtesting strategy which highlights the forecasting property of the Goodwin model on the considered data. Both the estimation and backtesting strategies can be used to assess the empirical improvement on any extension of the Goodwin model.

JEL Codes:: C15, E30, J20, E11

Keywords: Lodka-Volterra; Stochastic Differential Equation; Goodwin; Dynamical systems; Backtesting.

3.1 Introduction

It has been almost half a century since Goodwin (1967) developed a model of endogenous real growth cycles. Based on a simple and well known dynamic–the nonlinear Lotka-Volterra prey-predator model–Goodwin's model appeals in its simplicity and can be easily applied by a wide range of researchers in a variety of fields (physics, biology among others). In the late 1970s and 1980s, this research focused on relaxing one or more of the original model's assumptions and on adding new variables.[1] More recently, with the development of fast computing machines, which lower the costs of numerical simulation of continuous-time models, a large body of literature, especially in higher dimension, has emerged.[2] Although the literature is enriched by new theoretical extensions the empirical development has not been extensively explored. Currently, the best-known empirical research on Goodwin's model is perhaps that published by Harvie (2000).

In the year 2000, Harvie published a paper with mixed conclusions. On the one hand, using qualitative evaluation, Harvie acknowledged that the Goodwin model makes clear predictions on the interdependence of the employment rate and income distribution based on the clockwise behavior of the data over ten OECD countries. On the other hand, Harvie's findings for empirical estimation of the equilibrium point (the growth cycles' centers) and the cyclical periodicity for each country did not give satisfactory results. Harvie concluded by saying that further extension of the model should be explored in order to increase the reliability of the model's behavior. However, Grasselli Grasselli and Maheshwari (2016) showed that Harvie made reporting errors for the short term Phillips curve coefficients, thus destabilizing the conclusions. Furthermore, the findings of Grasselli and Maheshwari (2016) provide a more optimistic picture of the Lotka-Volterra-type model to fit empirical data, thus to explain the data's behavior.

In developing a strategy to estimate continuous-time models such as Lotka-Volerra's with low-frequency data, several potentially important caveats arise– see Section 3.3. Firstly, an intuitive way to tackle the estimation of such models would be to find a set of parameters that minimizes the distance of numerical deterministic simulations from the true observations.[3] Therefore, at each time, the difference between the true observation and the position of the estimated model is equal to the residual and is interpreted as being a measurement error. For example, if the observed value of the employment rate is not in the closed orbit of Goodwin's model, it is because this value has been wrongly assessed. Additionally, this type of estimation is also largely affected by the choice of

[1] See Desai (1973), Van der Ploeg (1985) or Van der Ploeg (1987) among others.

[2] See Keen (1995), Grasselli and Costa Lima (2012), Grasselli et al. (2014), Grasselli and Nguyen-Huu (2015), Nguyen-Huu and Costa-Lima (2014) among others.

[3] Numerical simulations rather than the explicit solution of the system are mentioned here since the latter is unlikely available.

the initial values. This is due to the fact that each simulation of the Goodwin model is a closed orbit,[4] thus indefinitely passing through the initial values, and therefore, the choice of the starting point will fundamentally change the outcome of the estimation. Another possible estimation strategy would be the maximum likelihood estimation. If one supposes that the Goodwin model is extended in a stochastic fashion, the system would then be made of stochastic differential equations (hereafter SDEs). Ideally, the exact transition density would be available to compute the maximum likelihood of the model. Unfortunately, the latter is known only in a few simple cases. When the solution is unknown, one can approach it by using a first-order approximation, but the lack of high-frequency data may generate an insufficient approximation, leading to biased estimation results.

In order to solve these problems, this article uses the technique developed by Pederson Pedersen (1995b), Pedersen (1995a) and Durham and Gallant (2002), which is commonly known as the simulated maximum likelihood estimation (hereafter SMLE). This technique is a promising alternative candidate for several reasons. First of all, it overcomes the problem of low-frequency data, since the simulated transition density converges towards the true transition density. Additionally, the estimation results are independent of the initial condition. Finally, by using SDEs rather than the deterministic counterpart, the model can explore the entire phase space, see Nguyen-Huu and Costa-Lima (2014).

After extending the Goodwin (1967) and the Van der Ploeg (1985) models to a stochastic framework, this paper estimates those models using the SMLE techniques with wage share and employment data in the United States (1948:Q1-2015:Q4). A preliminary analysis on the data shows two structural breaks located in 1984:Q1 and 2000:Q1. Furthermore, I show that van der Ploeg's extension model, in which the production sector is endowed with a CES production function, which allows for capital-labor substitution, is the best candidate to explain the data's behavior. A backtesting strategy based on out-of-sample error forecasts is proposed, with the aim of measuring the performance of the Goodwin model relative to a purely statistical vector autoregressive model (hereafter VAR).[5] Given the results, I show that stochastic Goodwin based models are a promising alternative to a VAR model for short term forecasting purposes. Although the Goodwin model is in most cases superior to the VAR, further improvements might be made in forecasting the employment rate–especially in crisis period–for example, by including the investment function in the same fashion than Keen (1995) model.

This paper is organized as follows: In Section 2, an overview of the deterministic Lotka-Volterra based model and its extension made Van der Ploeg (1985) is proposed, and

[4]Indeed the model is structurally unstable, see Goodwin (1967).

[5]VAR model, well known to be a non-economically based model and, also, for its forecasting ability, was chosen as the baseline model.

the extension of those models to stochastic differential equations is outlined. Section 3 introduces the framework for the estimation technique and a guideline of how the identification issue is tackled. Section 4 presents the data set and the treatment assumed in the paper and an analysis of the regularity of the data, turning to the results of the estimation of stochastic Goodwin models. The backtesting strategy is treated in Section 5. Finally, Section 6 offers concluding remarks and extensions.

3.2 The Lotka-Volterra Based Models

The aim of this section is threefold: (i) to introduce the Goodwin (1967) model and its extension made by Van der Ploeg (1985); (ii) to discuss Harvie's parameter estimates and; (iii) to extend those models, allowing for endogenous stochastic perturbations.

3.2.1 The Deterministic models

Goodwin Model (1967)

Goodwin (1967) introduced a growth cycle model of employment and wages based on a Lotka-Volterra predator-prey model. The predator-prey variables are the employment rate denoted by λ and the wage share, ω.[6] Assuming a Leontief production function, the model boils down to a two-dimensional system

$$\begin{cases} \dot{\omega} &= \omega \left(\phi(\lambda) - \alpha \right) \\ \dot{\lambda} &= \lambda \left(\frac{(1-\omega)}{\nu} - [\alpha + \beta + \delta] \right) \end{cases} \tag{3.1}$$

where the function $\phi(.)$ represents a short term Phillips curve, assumed to be increasing in λ, the employment rate. Parameters of the model and the values found in the literature are listed below, and when necessary, methodological issues are addressed.

The Productivity Growth, α

The parameter $\alpha > 0$ is the labor productivity growth and drives the deterministic growth of the output-to-labor ratio, a,

$$\frac{\dot{a}}{a} = \alpha,$$

[6]The full derivation of the model is presented in the appendix.

By defining the timeserie of a as the US GDP at constant prices over the employment level, Harvie (2000) estimated the following equation,

$$\log(a_t) = \log(a_0) + \alpha t + \varepsilon_t$$

and found an estimate for α, for the timeframe 1951-94, at 0.0111, while Grasselli and Maheshwari (2016) found 0.0155 for the period 1960-2010.

The Labor Force Growth, β

The labor force, N, is assumed to grow exponentially at a coefficient $\beta > 0$:

$$\frac{\dot{N}}{N} = \beta.$$

Using a similar method for labor productivity, Harvie (2000) estimated this parameter, for the period 1951-94, at 0.0206, while Grasselli and Maheshwari (2016) found 0.0165 for the period 1960-2010. .

The Depreciation of Capital, δ

As is standard, the stock of capital, K, is assumed to accumulate with respect to investment, I, and to depreciate at a constant rate, δ,

$$\dot{K} = I - \delta K.$$

Although in Harvie (2000) the depreciation rate of capital was not included in the model, in Grasselli and Maheshwari (2016), this parameter is assumed to be the mean value of the following timeseries

$$\delta^G := \frac{\text{Consumption of Fixed Capital in current prices}}{\text{Price deflator for gross fixed capital formation} \times \text{Net capital stock (2005)}}.$$

By doing so, they found a value for δ^G of 0.0521.[7] Using the above definition, δ depends on the level of the net capital stock, in particular on its initial value. In the database provided by AMECO, the level of capital is set using the rather strong assumption that the capital stock equals three times the nominal GDP in 1960 for every country. In other terms, to find the initial stock of capital, the methodology used by the AMECO is $K_{t_0=1960} = 3 \times GDP_{1960}$. Therefore, the level found for δ will change proportionally to the assumption regarding the initial capital.

[7] This computation is found using AMECO (the European Commission's annual macro-economic database from 1960 to 2010).

An alternative methodology to compute the depreciation rate is provided by the Penn World Table 8.1 database (hereafter PWT8).[8] In PWT8, the investment is divided into six classes, with each class having its own depreciation rate.[9] Therefore, the aggregated depreciation rate of capital of the whole economy will depend on what the capital is made of, and by consequence, the depreciation rate of capital will be time-variant. Using PWT8, an approximation of the depreciation rate of capital can by made by taking the mean value. This value would be 0.0376 for US data from 1951 to 2011. Note that, in PWT8, the initial capital stock is based on the assumption of an initial capital-to-output ratio methodology. More precisely, an initial amount is assigned for each of the six classes .[10] As previously mentioned, the initial value of each of the six classes and the path taken by the investment will influence the path of the depreciation of capital.

When using Bayesian techniques in dynamic stochastic general equilibrium modeling, the inference of the depreciation rate of capital suffers from a lack of identification and therefore cannot be estimated accurately. Therefore, δ is often assumed to be 0.025 per quarter, or put differently, roughly 10% on annual basis (for the Euro zone, see Smets and Wouters (2003); for the US, see Smets and Wouters (2007)).

No consensus emerges about the different methodologies used to find the accurate depreciation rate of capital. In such instances, for an annual frequency, one has three options: (i) 0.0521; (ii) 0.0376; (iii) 0.10. Since each of these values leads to different behaviors of the Goodwin model–especially for the employment rate–taking one of them may have a strong influence on the behavior of the estimation. Hence, I will let the data speak during the estimation of this parameter without any prior assumption on the level of the depreciation rate of capital.

van der Ploeg (1985)

Van der Ploeg (1985) relaxes the assumption that capital and labor cannot be substituted by endowing the economy with a CES production function,

$$Y = C\left[bK^{-\eta} + (1-b)(\lambda^L L)^{-\eta}\right]^{-\frac{1}{\eta}}$$

where $C > 0$ is the factor productivity and $b \in (0;1)$ is the share of capital. The short-run elasticity of substitution between capital and labor is given by $\sigma := \frac{1}{1+\eta}$. It

[8]Full details about the database are available in Feenstra et al. (2015)

[9]For the sake of clarity, structures (residential and non-residential) will have a depreciation rate of 2% while software will depreciate at 31.5% per year.

[10]The approach based on the steady state of the Solow model was considered and studied, but showed less stable results than linear regression techniques for a substantial number of countries. For further details, I refer to the Appendix C of Feenstra et al. (2015).

is worth recalling that the CES production function allows for three limit cases: (i) when $\eta \to +\infty$, one retrieves the Leontief production function; (ii) $\eta \to 0$ leads to the Cobb-Douglas production; (iii) if $\eta \to -1$ one recovers the linear production function.[11] Let us assume that the producer maximizes its profit given the wages .[12] It follows that the capital-to-output ratio is now endogenous. It is given by the first-order condition of profit maximization,

$$\nu(t) \quad := \quad \frac{K(t)}{Y(t)} = \frac{1}{C}\left(\frac{1-\omega(t)}{b}\right)^{-\frac{1}{\eta}}.$$

Van der Ploeg (1985) shows the important structural instability property of Goodwin model. Indeed, a minor modification in the parameters of the Lotka-Volterra model can lead to radical change in the quantitative behavior of the economic model. For instance, a small perturbation on the elasticity of substitution ($\sigma \cong 0$), the phase-portrait changes from a center to a stable focus: the model with the CES production technology. The reduced two-dimensional system is,[13]

$$\begin{cases} \frac{d\omega_t}{\omega_t} & = \left(\frac{\eta}{\eta+1}\right)\left[\phi(\lambda_t) - \alpha\right] dt \\ \frac{d\lambda_t}{\lambda_t} & = \left(Cb^{-1/\eta}(1-\omega_t)^{1+1/\eta} - (\delta+\beta+\alpha)\right) dt - \frac{1}{\eta}\left(\frac{d\omega_t}{\omega_t(1-\omega_t)}\right) \end{cases} \tag{3.2}$$

Since the assumptions of both models are similar, most of the parameters of the systems (3.1) and (3.2) look alike. The difference lies in the new parameters introduced by the CES production function. It also worth noting that one of the benefits of the van der Ploeg extension of the Goodwin model is that the trajectories taken by model (3.2) are less sensitive to small changes in the set of parameters than model (3.1).

Estimating those models parameter per parameter may lead to spurious results since certain key parameters for the dynamics, for instance δ, are very sensitive to the choice of the database and the methodology chosen to compute it. As a result, one can estimate the model as a whole with no assumptions for any of the parameters, especially for δ and ν. In what follows, the SMLE will be used to estimate the entire model. Before moving to this, however one needs to extend the model from deterministic to stochastic.

[11] As in Goodwin's seminal version, wages are set conformly to the short run Phillips curve. On the other hand, we confine ourselves to a real economy, so that the consumption price is normalized to 1.

[12] This minimal rationality argument is analogous to the assumption in Goodwin's model that the allocation of capital and labor is always at the diagonal of the (K, L)-plan, so that we have not only $Y = \min\left(\frac{K}{\nu}, aL\right)$ but also $Y = \frac{K}{\nu} = aL$.

[13] The full derivation is available in Appendix B.1. A slight change for simplicity and without any consequence has been made compared with Van der Ploeg (1985) since the labor productivity is not taken into account in the wage bargaining process.

3.2.2 The Stochastic Extensions

In order to introduce stochastic perturbations in the system, the following assumptions will be used

- *Assumption 1:* The labor productivity is defined as

$$\frac{da_t}{a_t} = \alpha dt - \sigma_1 dB_t^1$$

with B_t^1 a Brownian motion.

- *Assumption 2:* The real wages are set using a short-term stochastic Phillips curve,

$$\frac{dW_t}{W_t} = \Phi(\lambda_t)dt + \sigma_2 dB_t^2.$$

with B_t^2 a Brownian motion independent from B_t^1.

When applying both assumptions to the model (3.1), one retrieves[14]

$$\begin{cases} \frac{d\omega_t}{\omega_t} &= \left(\Phi(\lambda_t) - \alpha + \sigma_1^2\right) dt + \sigma_1 dB_t^1 + \sigma_2 dB_t^2 \\ \frac{d\lambda_t}{\lambda_t} &= \left[\frac{(1-\omega_t)}{\nu} - (\alpha+\beta+\delta) - \sigma_1^2\right] dt + \sigma_1 dB_t^1. \end{cases} \tag{3.3}$$

It is important to note that if $\sigma_2 = 0$ (ie. whenever the short-term Phillips curve is deterministic), then one recovers the model of Nguyen-Huu and Costa-Lima (2014). When applying these assumptions to the model (3.2), we get

$$\begin{cases} \frac{d\omega_t}{\omega_t} &= \left(\frac{\eta}{\eta+1}\right)\left\{\phi(\lambda_t) - \alpha - \frac{1}{2}\left(\frac{1-\eta}{(1+\eta)^2}(\sigma_1^2+\sigma_2^2) - \frac{\sigma_1^2}{\eta+1}\right)\frac{\sigma_2^2}{\eta+1}\right. \\ & \left. + \left(\frac{\sigma_1\eta}{1+\eta}\right)^2 + \left(\frac{\sigma_2}{1+\eta}\right)^2\right\} dt + \left(\frac{\eta}{\eta+1}\right)\sigma_1 dB_t^1 + \left(\frac{\eta}{\eta+1}\right)\sigma_2 dB_t^2 \\ \frac{d\lambda_t}{\lambda_t} &= \left(Cb^{-1/\eta}(1-\omega_t)^{1+1/\eta} - (\delta+\beta+\alpha)\right) dt \\ & - \left(\frac{\omega_t}{1-\omega_t}\right)^2\left(\frac{1}{1+\eta}\right)\left(\frac{\sigma_1^2+\sigma_2^2}{2}\right) dt - \left(\frac{1-\eta}{(1+\eta)^2}\frac{(\sigma_1^2+\sigma_2^2)}{2} - \frac{\sigma_1^2}{\eta+1}\right) dt \\ & + \left(\frac{\omega_t}{1-\omega_t}\right)\left\{-\eta\left(\frac{\sigma_1}{1+\eta}\right)^2 + \left(\frac{\sigma_2}{1+\eta}\right)^2\right\} dt + \left(\frac{\eta}{\eta+1}\right)^2 \sigma_1^2 dt + \left(\frac{\sigma_2}{1+\eta}\right)^2 dt \\ & + \left[\left(\frac{\omega_t}{1-\omega_t}\frac{1}{1+\eta}\right)\right]^2 (\sigma_1^2+\sigma_2^2)dt - \frac{1}{\eta}\left(\frac{d\omega_t}{\omega_t(1-\omega_t)}\right) + \sigma_1 dB_t^1 \end{cases} \tag{3.4}$$

Finally, it is worth mentioning that, if $\eta \to +\infty$, and if $A = 1/\nu$, model (3.4) boils down to model (3.3), and if in addition $\sigma_1 = \sigma_2 = 0$, those models are similar to the deterministic case, 3.1.

[14] See Appendix B.1 for the full derivation of the stochastic models.

3.3 The Estimation Technique

This section aims to present the methodology used for the estimation and address the identification issues.

3.3.1 Sketch of the SMLE

In previous attempts (Harvie (2000), Grasselli and Maheshwari (2016)) among others), the Goodwin model was estimated equation by equation. Each parameter was estimated separately using standard econometric tools such as an OLS, an error correction model or a vector error correction model. Mixed conclusions were drawn from those studies: in particular the long run equilibrium found was hardly consistent with phase space (ω, λ) shown by the data. In order to find better results, Harvie (2000) pointed out that certain theoretical extensions of the Goodwin model such as Desai (1973), aid in aligning the cyclical behavior given by the data. However, he also pointed out that the Goodwin model is econometrically challenging to estimate, and that additional extensions make the model more difficult to estimate empirically.

Instead of proposing the estimation of new theoretical extensions of the Goodwin model, this Section aims at providing another estimation approach for such models. Rather than estimating the model parameter by parameter, I directly estimate the whole nonlinear dynamical system. The most obvious benefit of the approach is that the estimation of δ and hence the whole model, does not rely on the assumption of the level of the initial capital stock made by the database under consideration as previously mentioned. The estimation will instead be based on the estimation of multidimensional SDEs.

SDEs are wildly used in finance, pricing theory (see Black and Scholes (1973)), yield curve models (see the HJM model from Heath, Jarrow and Heath et al. (1990)), and algorithmic trading among others. Financial markets, rather than macroeconomics, are more suitable for a SDEs setting because data is often available at a high frequency.[15] Nonetheless, tools to infer the ability of SDEs to cope with lower frequency data have been developed.

The estimation methodology is borrowed from Durham and Gallant (2002) and is extended to the multivariate framework.[16] Let us consider a reduced-form SDE on the

[15]The standard time mesh can be some fractions of a second. Such high-frequency data is not available in macroeconomics where the time mesh is often a quarter, or perhaps a year.

[16]What follows is a sketch of the methodology, an extensive explanation is available in Appendix B.2.

probability space $(\Omega, \mathcal{F}, \mathbb{P})$ of the form

$$\begin{cases} dX_t &= f(X_t)dt + g(X_t)dB_t, \\ X_{t_0} &= X_0. \end{cases}$$

Where $X_t \in \mathbb{R}^n$ is the state variable vector, B_t is a d-dimensional Brownian motion, $f : \mathbb{R}^n \to \mathbb{R}^n$ is the drift of the process and $g : \mathbb{R}^n \to \mathbb{R}^{n \times d}$ is the diffusion. For the sake of clarity, $X_t = (\omega_t, \lambda_t)^T$, where T is the transpose operator.

Ideally, to compute the maximum likelihood estimation, one should know the transition density. Because analytic solutions are rarely available in practical situations, the transition densities must be approximated numerically. Therefore, numerical methods are required to approximate their solutions. In what follows, the Euler-Maruyama scheme is used (see Kloeden and Platen (1992)). On the one hand, the Euler-Maruyama is computationally intensive in minimizing the error of the numerical methods, but on the other hand, this scheme is computationally feasible at all times in multivariate framework. For instance, if one uses the scheme proposed by Jimenez, Jimenez et al. (1999), one should keep in mind the authors' caution: "... this numerical scheme is not always computational feasible since it can fail for SDE for which the Jacobian matrix $J_f^{-1}(X)$ is singular or ill-conditioned in at least a point" (Jimenez et al. (1999), p.593). For the sake of clarity, the Euler-Maruyama scheme is

$$\tilde{X}_{i+1} = \tilde{X}_i + f(\tilde{X}_i)\delta + g(\tilde{X}_i)\delta^{1/2}\varepsilon_i$$

where $\delta = t_{i+1} - t_i$, $\varepsilon_i \sim \mathcal{N}(0, 1)$, and $\tilde{X}$ is the approximated counterpart of X. Under some mild assumptions, it can be shown that this approximation converges to the true maximum likelihood. Nevertheless, the approximation may not be sufficiently accurate for the sampling frequencies, especially for macroeconomic data. In Durham and Gallant (2002), the proposed methodology is named the simulated maximum likelihood estimation (hereafter SMLE). The general idea is to obtain the true transition probability, $p(x_t, t; x_s, s)$. Using the Euler-Maruyama scheme, one can approximate the true transition density by $p^{(1)}(x_t, t; x_s, s)$. As previously mentioned, the frequency is too low to provide a good convergence to the true maximum likelihood estimation. An idea is to generate a subinterval $s = \tau_1 < \ldots < \tau_M = t$, so that the random variable is sufficiently accurate at each subinterval. The vector $(X(\tau_2), \ldots, X(\tau_{M-1}))$ is therefore unobserved and should be simulated by a Brownian bridge. Because the process is Markovian, one obtains

$$\begin{aligned} p(x_t, t; x_s, s) &\approx p^{(M)}(x_y, t; x_s, s) \\ &:= \int \prod_{m=0}^{M-1} p^{(1)}(u_{m+1}, \tau_{m+1}; u_m, \tau_m) \\ &\quad \times d\lambda^{Leb}(u_1, \ldots, u_{M-1}) \end{aligned}$$

where λ^{Leb} is the Lebesgue measure. The integral can be evaluated using Monte Carlo integration. By doing so, one obtains the simulated transition probability, $p^{(M,K)}(x_y, t; x_s, s)$, where K is the Monte-Carlo parameter. By repeating this operation for each transition of the dataset, I compute the simulated likelihood.

3.3.2 Identification Issues

In order to infer the model (3.3) with the Leontief production function, it is necessary to identify the parameters. Because, for instance, the estimation procedure does not distinguish between β and δ, one needs to rewrite the model to make it suitable for the estimation. The following model will be estimated:

$$\begin{cases} \frac{d\omega_t}{\omega_t} &= \left(\Phi^*(\lambda_t) - \phi_0 + \sigma_1^2\right) dt + \sigma_1 dB_t^1 + \sigma_2 dB_t^2 \\ \frac{d\lambda_t}{\lambda_t} &= \left[\frac{(1-\omega_t)}{\psi_0} - \psi_1 - \sigma_1^2\right] dt + \sigma_1 dB_t^1 \end{cases}$$

where $\Phi^*(\lambda_t)$ is the short term Phillips curve without constant; and ϕ_0 is the constant of the short term Phillips curve minus the labor productivity. ψ_0 remains the capital-to-output ratio, while ψ_1 is the combined parameter of $(\alpha + \beta + \delta)$. Turning to the model (3.4), the CES production function, the same specifications for ϕ_0 and ψ_1 are made. The only difference is that for $Cb^{-1/\eta}$, the idiosyncratic effect of C and b cannot be distinguish; for the estimation it will be denoted by C_b. It is worth mentioning that parameter η, which controls the substitution between capital and labor, is well defined since it will weight the influence of the wage share dynamic on the employment rate dynamic.

$$\begin{cases} \frac{d\omega_t}{\omega_t} &= \left(\frac{\eta}{\eta+1}\right)\left\{\Phi^*(\lambda_t) - \phi_0 - \frac{1}{2}\left(\frac{1-\eta}{(1+\eta)^2}(\sigma_1^2+\sigma_2^2) - \frac{\sigma_1^2}{\eta+1}\right)\frac{\sigma_2^2}{\eta+1}\right. \\ &\left. + \left(\frac{\sigma_1\eta}{1+\eta}\right)^2 + \left(\frac{\sigma_2}{1+\eta}\right)^2\right\} dt + \left(\frac{\eta}{\eta+1}\right)\sigma_1 dB_t^1 + \left(\frac{\eta}{\eta+1}\right)\sigma_2 dB_t^2 \\ \frac{d\lambda_t}{\lambda_t} &= \left(C_b(1-\omega_t)^{1+1/\eta} - \psi_1\right) dt \\ &- \left(\frac{\omega_t}{1-\omega_t}\right)^2\left(\frac{1}{1+\eta}\right)\left(\frac{\sigma_1^2+\sigma_2^2}{2}\right) dt - \left(\frac{1-\eta}{(1+\eta)^2}\frac{(\sigma_1^2+\sigma_2^2)}{2} - \frac{\sigma_1^2}{\eta+1}\right) dt \\ &+ \left(\frac{\omega_t}{1-\omega_t}\right)\left\{-\eta\left(\frac{\sigma_1}{1+\eta}\right)^2 + \left(\frac{\sigma_2}{1+\eta}\right)^2\right\} dt + \left(\frac{\eta}{\eta+1}\right)^2\sigma_1^2 dt + \left(\frac{\sigma_2}{1+\eta}\right)^2 dt \\ &+ \left[\left(\frac{\omega_t}{1-\omega_t}\frac{1}{1+\eta}\right)\right]^2(\sigma_1^2+\sigma_2^2)dt - \frac{1}{\eta}\left(\frac{d\omega_t}{\omega_t(1-\omega_t)}\right) + \sigma_1 dB_t^1 \end{cases}$$

3.4 Data and the Estimation Results

This section presents the data sources and methodology to construct the phase variables (ω, λ) are discussed. Secondly, data's properties are examined. Finally, different specifications for the short term Phillips curve are derived.

3.4.1 Data Construction and Preliminary Analysis

Data used for the estimation are taken from two main sources: (i) U.S. Bureau of Economic Analysis; and (ii) U.S. Bureau of Labor Statistics. The frequency of the data is quarterly and runs from 1948:Q1 to 2015:Q4. The two main variables used are the labor share, ω, and the employment rate, λ.The employment rate is defined as:[17]

$$\lambda := \frac{\text{Total Employment}}{\text{Total Labor Force}}.$$

The wage share is

$$\omega = \frac{\left(1 + \frac{\text{Self Employed}}{\text{Total Employees}}\right)\text{CE}}{\text{GDP at factor cost}},$$

where CE stands for the compensation of employees, which is the total gross (pre-tax) wage paid by employers to employees within a single quarter. Although a large part of the total wages earned in the economy is determined by the compensation of employees, a substantial amount is located in the gross operating surplus (hereafter GOS) due to the self-employed (while it represented more than 18% of the total workers in 2015, this category dropped down to 8% in 2015).[18] In order to have a more realistic measure of the weight of the wage in the economy, one can make the assumption that the self-employed, on average, earn as much as employees. Taking this assumption, I

[17] The definition of the labor share is similar to Harvie (2000).

[18] This idea is discussed extensively in Mohun and Veneziani (2006).

add a proportional share, representing the wages earned by self-employed, to the CE (Self Employed/Total Employees).[19] Turning to the denominator of ω, GDP is measured at factor cost. Since the income approach of the GDP at market price is

$$\begin{aligned} \text{GDP (market price)} &= \text{CE} + \text{GOS} + \text{T-S} \\ \text{GDP (market price)} - \text{T-S} &= \text{CE} + \text{GOS} \\ &= \text{GDP (factor cost)} \end{aligned}$$

where GOS can be read as the EBITDA (earnings before interests, taxes, depreciation, and amortization), and T-S is the net taxes on products and imports.[20] Figure 3.1

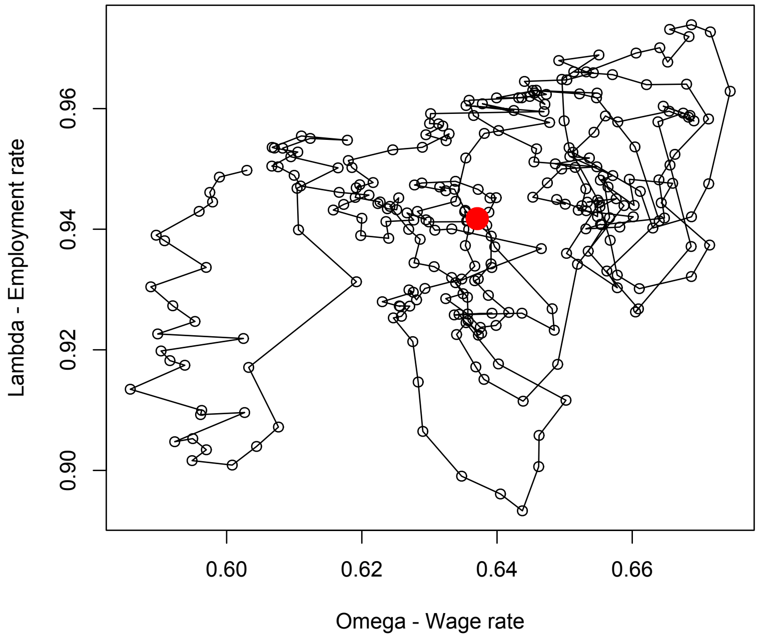

FIGURE 3.1: The empirical phase portrait of the variable (ω, λ). In red, the empirical mean of the state variables.

Source: BEA – US data from 1948:Q1 to 2015:Q2

represents the empirical phase portrait of the state variables ω, on the x-axis and λ, on the y-axis. Using qualitative evaluation on similar data sets, Solow (1990), Harvie (2000) and Mohun and Veneziani (2006), showed that the data have a clockwise behavior, as would be expected from Goodwin's theory. It is worth noting that in the left part of the quadrant, the last cycle, that started in 2007:Q4, is the most at odds with previous cycles. This inconsistency is mainly due to current wage shares being lower than those that were explored over the sample. Also, one can note, from a qualitative perspective, SDEs should provide a feasible modeling if one wants to replicate such kind of trajectory. The red dot in Figure 3.1 represents the $(x, y)-$coordinate of the empirical means of the

[19]This methodology is borrowed from Grasselli and Maheshwari (2016).
[20]For the sake of clarity, it can be seen as the V.A.T., or subsidies such as environmental externalities.

phase space. Despite that this red dot is, qualitatively, at the center of the portrait, it seems that multiple cycles are represented in Figure 3.1, subsection 3.4.2 shows some evidences with descriptive statistics.

3.4.2 Evidence of structural changes

There is a large body of recent macroeconomic literature that focuses on changes in the relationship (causality, dependency, explanatory strength, etc.) between macroeconomic variables such as GDP, oil price, consumption, investment among others. Those changes are referred as structural breaks. Various methodologies could be considered, for instance, Kim and Nelson (1999) and Perez-Quiros and McConnell (2000) have used a volatility reduction Markov switching model and independently found a structural break at the date 1984:Q1. Later, using an alternative approach, Stock and Watson (2003) confirms that the volatility of macroeconomic variables has declined at the aftermath of the FED's aggressive response to inflation, during the Volcker era, that was credited to end the United States' stagflation crisis of the 1970s.

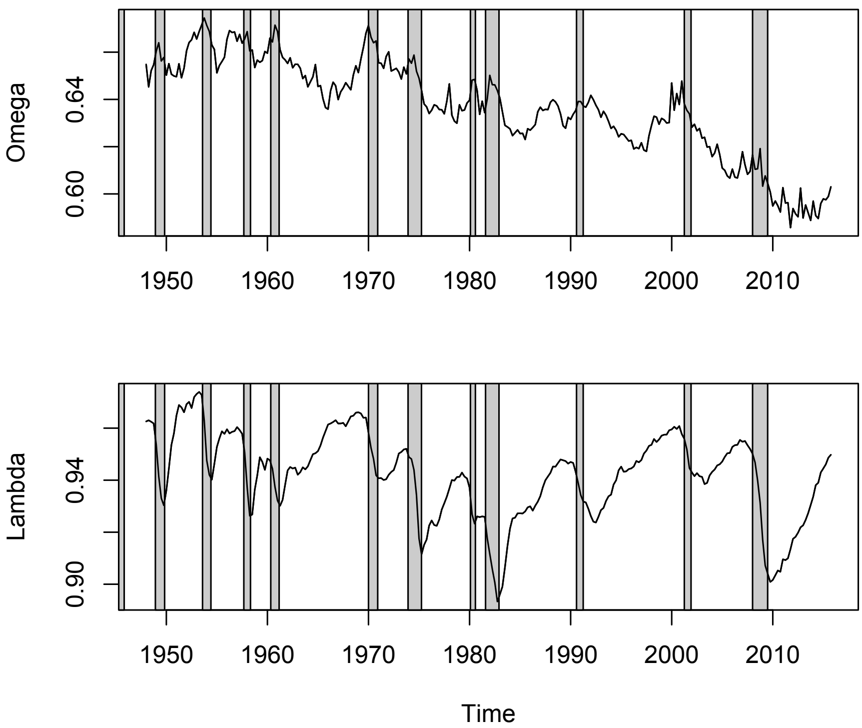

FIGURE 3.2: Timeseries of the wage share (top) and the employment rate (bottom). The shaded grey represents NBER recessions.

Source: BEA – US data from 1948:Q1 to 2015:Q2

Figure 3.2 shows the evolution of the wage share (top) and the employment rate (bottom) over time. The shaded areas refer to the NBER recession periods. Those recession periods are highlighted since they somehow represent the end of the Lotka-Voletrra cycle symbolized by the drop in the employment rate and, hence, may be the premise of a new cycle era. While the wage share qualitatively shows a downward bending long term trend that plateaued over the last five to ten years, interestingly, the employment rate shows first a decreasing trend until mid-1980s' and, then, an upward trend until the subprime mortgage crisis (the last grey-shaded area). For the sake of descriptive

Sub-periods	Mean of ω	St.dev. of ω	Mean of λ	St. dev. of λ
1948:Q1 - 1984:Q1	0.652	0.011	0.945	0.017
1984:Q2 - 2000:Q1	0.631	0.007	0.940	0.010
2000:Q2 - 2015:Q4	0.610	0.015	0.937	0.018

TABLE 3.1: First and second empirical moment of (ω, λ) for given sub-periods.

statistics, table 3.1 presents the empirical mean values and the standard deviations of the state variables over different time frame.[21] The results show the decline of volatility, as documented in Stock and Watson (2003), over the first (1948:Q1-1984:Q1) and the second (1984:Q2-2000:Q1) sub-period and also the downward shift of the mean of the state variable ω that loses almost two standard deviations from the first to the second sub-period. On the other hand, the last sub-period shows that, on average, wage-to-GDP ratio and the employment rate levels are lower than previous sub-periods with a return to a relatively high volatility era equivalent to sub-period one.

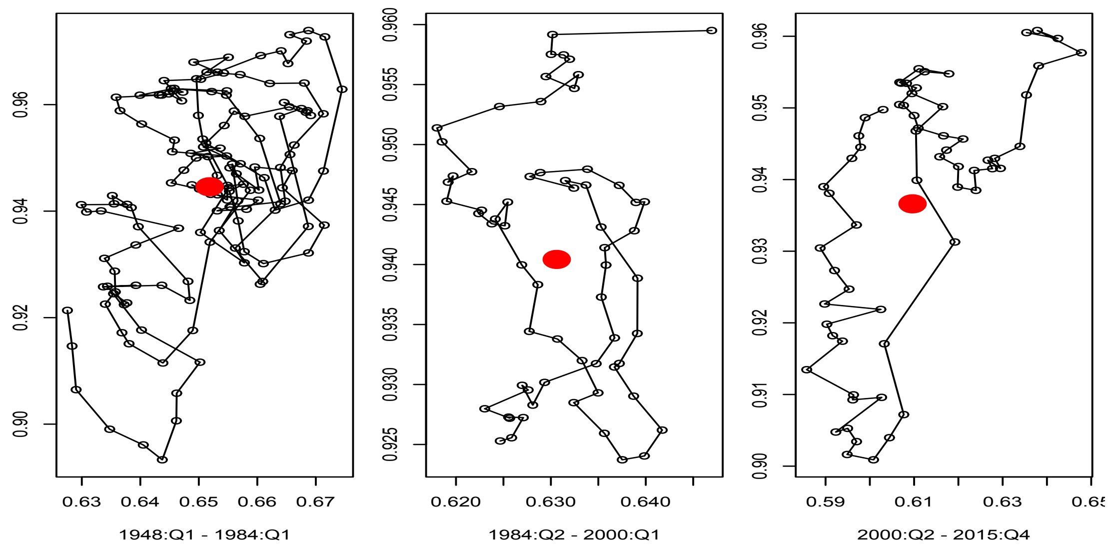

FIGURE 3.3: Phase space of the three sub-periods.

Source: BEA – US data from 1948:Q1 to 2015:Q2

As illustrated by Figure 3.3, the observations around the empirical mean values, represented by a red dot, of each sub-period show less dispersion that in Figure 3.1. Therefore,

[21]The motivation behind the selected time frame will be discussed shortly.

qualitatively, and as shown in Mohun and Veneziani (2006), one can conclude that various cycles with different frequencies and equilibrium can be found in the data.

For the sake of completeness, Figure 3.4 shows each sub-period on the same scale with different colors: (i) black for the period 1948:Q1-1984:Q1; (ii) the period 1984:Q2-2000Q1 is represented with the color blue; and (iii) red illustrates the last period 2000:Q2-2015:Q4. Dots represent the empirical mean coordinate of their respective colors. The downward sloping trend of the wage share over time in the empirical observation is well illustrated in this graph since, along the sub-periods, the scatter plots, as well as the empirical means, are heading from the north west towards the south west of the phase diagram.

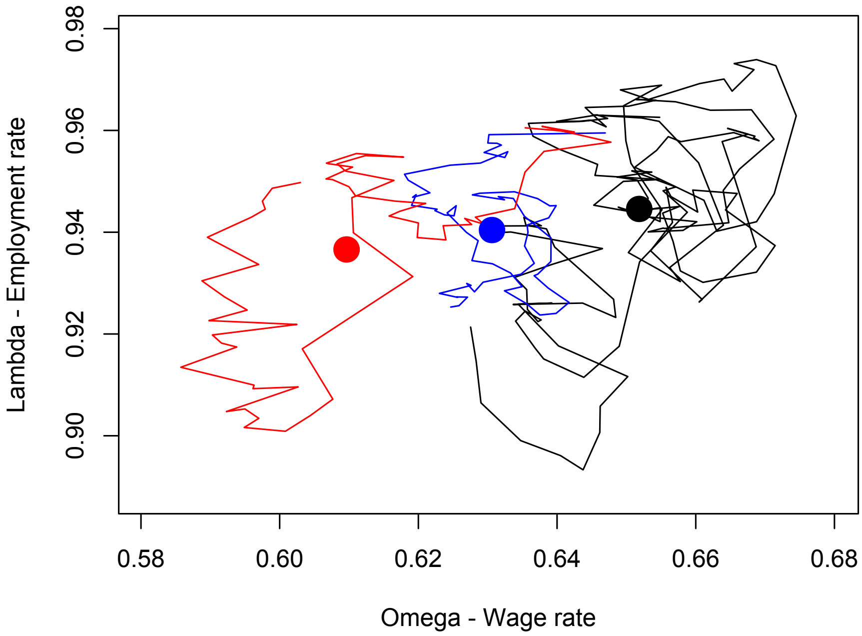

FIGURE 3.4: Phase space of the three sub-periods.

Source: BEA – US data from 1948:Q1 to 2015:Q2

3.4.3 Short Term Phillips Curve

A degree of freedom for the global behavior of the dynamic and allowed by Goodwin (1967) lies in the short term Phillips cure. Perhaps, in Goodwin's estimation framework, previous attempts to estimate the phenomenological behavioral function were essentially made using OLS, see Harvie (2000), with the aim of estimating:

$$\frac{\dot{w}}{w} = \phi(\lambda). \tag{3.5}$$

Nevertheless, using such a framework to estimate differential equation such as the Phillips curve may lead to spurious results. For simplicity, consider that the time between t and $t+1$ is one year, w_t is the real wage, and λ_t is the employment rate. Additionally, suppose that using quarterly data, the following linear regression is well specified (meaning that the residuals pass standard tests)

$$\log\left(\frac{w_{t+1/4}}{w_t}\right) = \alpha_0 + \alpha_1\lambda_t + \varepsilon_{t+1/4}.$$

Taking the deterministic part, one can rewrite the same equation as

$$\int_t^{t+1/4} \frac{dw_t}{w_t} = \alpha_0 + \alpha_1\lambda_t,$$

and by taking the first derivative with respect to t, we are led to a differential equation with delay:

$$\frac{\dot{w}_{t+1/4}}{w_{t+1/4}} = \frac{\dot{w}_t}{w_t} + \alpha_1\dot{\lambda}_t,$$

which involves the theory of delay differential equations. Therefore, if one wants to infer the short term continuous Phillips curve using OLS, one does not obtain the desired equation 3.5 because of the discretization biais.[22]

The estimation methodology proposed in this paper allows the inference of the short term Phillips curve from its original specification. Since Goodwin (1967) allows for multiple varieties of that phenomenological function, the estimations are made with respect to each of those three specifications:

$$\phi(\lambda) = \phi_0 + \phi_1\lambda, \tag{3.6}$$
$$= \phi_0 + \frac{\phi_1}{(1-\lambda)}, \tag{3.7}$$
$$= \phi_0 + \frac{\phi_2}{(1-\lambda)^2}. \tag{3.8}$$

One can see that the variation of the wages with respect to the employment rate will be amplified along the short term Phillips curve, 3.6, 3.7 and 3.8.

3.4.4 Estimation Results

The aim of this section is twofold: (i) to present the fitting of the estimation of the whole model, this has to be understood as a proof of concept for the SMLE approach; and (ii) to test for structural breaks in the cycle.

[22] I refer the reader to Appendix B.4 for the test on some data generating process.

3.4.5 The Fitting

As a preliminary exercise, this section presents the estimation of models (3.3) and (3.4) with each of the previously specified short term Phillips curve (3.6), (3.7) and (3.8) over the whole sample (1948:Q1-2015:Q4). The quality of the fitting will be measured by the well-known AIC criterion (see. Akaike (1973)). It is defined as

$$\text{AIC} := -2 \times \log(\text{Likelihood}) + 2 \times \text{Number of parameters}.$$

This measure allows for the assessment of the relative quality of statistical models for given datasets. This model selection procedure results in a trade off between the goodness of fit–the log-likelihood–and the number of estimated parameters. The model that has the minimal value for the AIC criterion would be qualitatively the best to fit the data.In the following, each likelihood is computed using first $M = 8$ and $K = 8$ and second with $M = 16$ and $K = 126$. This two-stage procedure enables to reach the optimum faster.

	Leontief (3.3)	CES (3.4)
Short term Phillips curve (3.6)	-4307.01	-4394.54
Short term Phillips curve (3.7)	-4311.35	**-4396.47**
Short term Phillips curve (3.8)	-4306.84	-4392.38

TABLE 3.2: The AIC values of the Leontief and the CES models.

According to the AIC criterion, the result provided by table B.2 is twofold: (i) the model with the CES production function gives a better statistical performance than its limit case, the Leontief production function, and the result holds for each short term Phillips curve; (ii) for both the Leontief and the CES production functions, the short term Phillips curve that gives the best statistical fit is given by equation (3.7). The Van der Ploeg (1985) extension of the Goodwin model significantly increases the reliability of the model's behavior. Additionally, as proved by van der Ploeg, if one wants to use the estimates for prospective scenarios, the trajectory taken by the model with the CES function will not be significantly altered by a small change of parameter. Furthermore, it is impossible to estimate the true parameters because of the confidence interval of each estimate. Therefore, less unstable models are recommended.

3.4.6 The Parameter Estimates

In aiming to test the economic reliability, over the whole sample, of the parametric estimation strategy with all the previously specified short term Phillips curve, this section displays the estimates found for model (3.3), with a Leontief production function and model (3.4), the CES production function counterpart. Results will then be discussed.

3.4.6.1 The Leontief Production Function

	ϕ_1	ϕ_0	ψ_0	ψ_1	σ_1	σ_2
PC (3.6)	0.2741 (0.1268)	0.2593 (0.1194)	6.618 (1.9203)	0.0549 (0.0159)	0.0081 (0.0003)	0.0007 (0.0167)
PC (3.7)	0.000874 (0.0004)	0.01734 (0.0075)	6.6192 (1.9239)	0.05492 (0.0159)	0.0081 (0.0003)	0.0167 (0.0007)
PC (3.8)	$1.395e-05$ ($1.959e{-}05$)	0.0085 (0.0054)	6.583 (1.912)	0.0552 (0.01605)	0.0081 (0.0003)	0.0007 (0.0167)

TABLE 3.3: The parameter estimates of model 3.3 and the standard deviation below.

Over the three specifications of the short term Phillips curve, the estimates of ψ_0, ψ_1, σ_1, and σ_2 displayed in table 3.3 are relatively similar. Interestingly, the capital-to-output constant ratio, ψ_0, is estimated to be 6.6. This value is at odds with previous findings in the sense that it appears overestimated. On the other hand, ψ_1 displays a value of about 5.5%. Since it represents the compounded value of $(\alpha + \beta + \delta)$, and by taking reasonable values for α and β previously discussed, this would mean that δ is inferred to be approximately 2.4%, lower than what as been previously discussed. As previously mentioned, this value is the result of the estimation on the whole sample. However, in the case of structural change (or nonlinearity) in the timeserie, this estimate may lead to spurious results as it may be in this case here. This problem will be addressed shortly. One can note that the parameter estimates for each of the short term Phillips curves are positive and significant.

3.4.6.2 The CES Production Function

	ϕ_1	ϕ_0	C_b	ψ_1	η	σ_1	σ_2
PC (3.6)	0.2975 (0.1030)	0.2815 (0.0971)	0.1289 (0.0503)	0.0404 (0.0156)	6.0478 (1.0762)	0.01027 (0.0008)	0.01330 (0.0008)
PC (3.7)	0.0010 (0.00031)	0.0195 (0.0061)	0.1261 (0.051)	0.0395 (0.0157)	6.003 (1.0645)	0.0103 (0.0008)	0.0133 (0.0008)
PC (3.8)	$2.242e-05$ ($1.21e{-}05$)	0.0098 (0.0047)	0.1282 (0.0503)	0.0401 (0.0158)	5.9910 (1.056)	0.0103 (0.0008)	0.0133 (0.0008)

TABLE 3.4: The parameters estimate of model (3.4) and the standard deviation below.

The estimated parameters revealed in table B.4 show that the ϕ_0 and ϕ_1, namely the short term Phillips curve parameters, are similar to the previous estimates. The remaining parameters $(C_b, \psi_1, \eta, \sigma_1, \sigma_2)$ are relatively close to each other. The major difference is in the elasticity of substitution between capital and labor; it is approximately equal to 0.1428 ($\approx 1/(\eta + 1)$).

3.4.7 Structural Breaks

Tests for structural breaks have been extensively studied in timeseries analysis. Back in 1960, to the best of my knowledge, Chow (1960) published the first paper that tested parameter change in a linear regression. Recently, Bai and Perron (2003) developed more sophisticated techniques for multiple breaks in the timeseries. In this paper, with the SMLE estimation framework, I minimize the following criterion in order to detect a structural break:

$$\text{BIC} := -2\log(\text{Likelihood}) + \log(\text{Sample Size}) \times \text{Number of parameters}.$$

This model selection criterion was first introduced by Schwarz (1978) and is called Bayesian information criterion, or BIC. Compared to the AIC, the BIC penalizes more the number of parameters by putting more weight, thus it prevents for overfitting. This choice is motivated by the fact that a structural break will increase the number of parameters, since they will be time-varying. Thus, it will be harder to detect a break using this technique, unless it significantly improves the likelihood of the estimation on the whole sample.

Number of structural break(s)	0	1	2
BIC	-4279.37	-4279.00	**-4309.449**
Located break(s)	x	2008:Q2	1984:Q1 and 2000:Q1

TABLE 3.5: BIC criterion of model 3.3 with the short term Phillips curve 3.5 for 0, 1 and 2 structural breaks and their location.

Table 3.5 presents the locations where the BIC criterion is minimal for zero, one, and two structural breaks of the model (3.3) with the short term Phillips curve (3.5). The model with one structural break shows similar results than the model with no break. However, the model with two structural breaks minimizes the BIC criterion despite its strong penalty on the number of parameters. Interestingly, the breaks are located in 1984:Q1, which is consistent to the break date discussed in subsection 3.4.2, and 2000:Q1. The last break date represents a change that occurred slightly before the dot-com bubble crises.

	Leontief (3.3)	CES (3.4)
Short term Phillips curve (3.6)	-4309.449 (-4279.37)	-4381.466 (-4362.301)
Short term Phillips curve (3.7)	-4307.207 (-4281.988)	**-4383.706** (-4367.012)
Short term Phillips curve (3.8)	-4306.958 (-4279.203)	-4381.626 (-4360.141)

TABLE 3.6: The BIC values for the model with two breaks and in parenthesis below the BIC value for the model with no break presented above.

Table 3.6 presents the BIC values for models (3.3) and (3.4) with all the short term Phillips curve. Below each value (in parenthesis) the BIC value of the counterpart model with no break is displayed for comparison. Three conclusions can be drawn out: First, for each attempt, the model with two break points explains the data more accurately in regards to the counterpart model without break. Second, for each specification of the short term Phillips curve, the model endowed with a CES production technology shows a better fitting according to the BIC criteria. This result confirms that the model endowed with CES production function is better in explaining the dynamics. Finally, the model with a behavioral function of the wages given by equation 3.7 gives a better fitting for both the Leontief and CES technology. As in the case with no break, Table 3.6 shows that model (3.4) with equation (3.7) show in general the best fitting.

	ϕ_1	ϕ_0	ψ_0	ψ_1	σ_1	σ_2
1948:Q1 - 1984:Q1	0.0008	0.0165	1.7813	0.1966	0.0093	0.0175
	(0.00048)	(0.0101)	(0.4183)	(0.0458)	(0.0005)	(0.0010)
1984:Q2 - 2000:Q1	0.00064	0.0107	2.6528	0.1368	0.0038	0.0092
	(0.0008)	(0.0134)	(1.0352)	(0.0544)	(0.0003)	(0.0008)
2000:Q2 - 2015:Q4	−0.0002	0.0010	3.0489	0.1285	0.0068	0.01905
	(0.0011)	(0.0201)	(1.0162)	(0.0427)	(0.0006)	(0.0169)

TABLE 3.7: The parameter estimates of model (3.3) with a short term Phillips curve (3.7) and the standard deviation below over the different sub-periods.

Estimated parameters of model (3.3) with a short term Phillips curve (3.7) are displayed in table (3.7). Over the different sub-periods, three main differences with table (B.3) are worth noting. First, the short term Phillips curve parameters are less significant. Indeed, while former parameters were significant at the 95% level, the new parameters found are at most significant up to a 90% level. Also, the sign of the response of wages to the employment rate is reversed compared to what one would expect, however this parameter is strongly insignificant. Therefore, a conclusion would be that the nonlinearity for the short term Phillips curve could have changed in the last sub-period. On the other hand, the values found for the capital-to-output ratio, ψ_0, are almost in line with the literature for each attempt. Remember, without any prior assumption on the level of the capital-to-output ratio, the previous finding was approximately 6.6. Figure 3.5 shows the variation of the ratio over the period 1950-2013 according to the PWT8 database. Within sub-period 1984:Q1-2000:Q1, this ratio oscillates around 2.8, while the value found is 2.6528 and for the sub-period 2000:Q2-2015:Q4 the latter fluctuates around 3, close to the estimated value 3.0489. The change in that estimate generates an adjustment of ψ_1, in line with the literature, compared with the estimates of the model with no break. This value increases from 5% to approximately 13% that implies a depreciation rate of capital of about 10% per year. Finally, parameters σ_1 and σ_2, namely the volatility of the data, reflect the great moderation era by showing a lower volatility for the second sub-period with respect to the others. It is worth mentioning that the oscillation shown by Figure

FIGURE 3.5: The capital-to-output ratio of the United-States - 1950-2013.

Source: PWT8

3.5 paves the road to the rationale that CES technology, and its time-varying capital-to-output ratio, seems to be a better candidate at explaining the data than Leontief technology.

Dates	ϕ_1	ϕ_0	C_b	ψ_1	η	σ_1	σ_2
48:Q1-84:Q1	0.0007 (0.0004)	0.0158 (0.0082)	0.5139 (0.1528)	0.1416 (0.0421)	4.2298 (0.9525)	0.0126 (0.0016)	0.0130 (0.0012)
84:Q2-00:Q1	0.0008 (0.0006)	0.0133 (0.0113)	0.3934 (0.1589)	0.1254 (0.0517)	7.8096 (2.1113)	0.0043 (0.0004)	0.0076 (0.0008)
00:Q2-15:Q4	$-2.15e-05$ $(1.909e{-}03)$	0.0041 (0.0191)	0.3442 (0.1158)	0.1305 (0.0434)	25.42 (14.85)	0.0073 (0.0007)	0.0180 (0.0017)

TABLE 3.8: The parameters estimate of model (3.4) and the standard deviation below.

Table 3.8 presents the estimated parameters for model (3.4) with the short term Phillips curve (3.7). It shows similar conclusions for the estimates of the short term Phillips than for the Leontief counterpart. One can note that the estimates of ϕ_0 are similar for the first and second sub-periods. Also, the η parameters keep increasing along the sub-periods meaning that capital-labor substitution keeps decreasing along the time, estimates of the elasticity of substitution are chronologically 19.12%, 11.35%, and 7.78%. Likewise the previous estimates of the Leontief production technology, the estimates of the volatility parameters present similar conclusion in the CES case.

3.5 The Backtesting

This section shows the methodology that is used to implement the backtesting. Moreover, the results for the no break case and the one with two breaks are displayed and analyzed.

3.5.1 The Methodology

In seeking to measure the performance of the model, the proposed methodology was inspired by Kilian and Vigfusson (2013) among others. The underlying concept is based on out-of-sample error forecast. The structure for the backtesting strategy is outlined in the following: Suppose that one has a dataset starting at T_0 and ending at T_1,

- Step 1: Choose a date between T_0 to T_1, say T^*.
- Step 2: Make an estimation of the model from T_0 to T^*.
- Step 3: Taking the deterministic form of the model into consideration, run a simulation for h-periods.
- Step 4: Compare, using distance d, the simulated value to the realized value obtained in Step 3.
- Step 5: Repeat steps 2-4 by increasing the T^* by one period through the end of the sample.

To measure the performance of the forecast, assume the following distance:

$$d_\gamma(x, y) = |x - y|^\gamma.$$

The h-period-ahead forecast performance will be evaluated in the following way: by considering m periods,

$$D_h^M = \sum_{j=1}^{m} d_\gamma(x_{i_h}^M, x_{i_h}^t),$$

where $x_{i_h}^M$ is the h-period-ahead forecast of the model M estimated from the beginning of the sample to the i-th period and, $x_{i_h}^t$ is the realized value at date $i + h$. In the following exercise, $m = 100$ (or 25 years),[23] $\gamma = 1$,[24] and $h = 1, \ldots, 8$.[25] To test the relative performance of the forecast, a VAR will be used as the benchmark model. Using the AIC criterion, the optimal lag for the benchmark VAR model is 3.

3.5.2 The Results for models with no break

	Leontief		CES	
	ω	λ	ω	λ
h = 1	**0.9399747**	1.276043	**0.9446652**	3.219161
h = 2	**0.8701238**	1.230077	**0.8818391**	3.326414
h = 3	**0.8311719**	1.245011	**0.8556275**	3.397210
h = 4	**0.8023928**	1.212495	**0.8545608**	3.316807
h = 5	**0.8230414**	1.212181	**0.8831599**	3.285173
h = 6	**0.8463071**	1.219925	**0.9314075**	3.281554
h = 7	**0.8645347**	1.259291	**0.9601584**	3.357697
h = 8	**0.8580543**	1.286638	**0.9901404**	3.399121

TABLE 3.9: Relative performance of the model against a VAR with the short term Phillips curve (3.6).

Table 3.9 shows the forecasting performance of the Goodwin models under consideration, with a linear short term Phillips curve against a VAR. Values below unity, in bold, indicates that the underlying model is globally performing better than a VAR for the state variable named by the column. For instance, 0.9399747 means that the 1-period-ahead forecast of the Goodwin model endowed with a Leontief production function with a linear Phillips curve performs roughly 6% better than a VAR model for the state variable ω.

The results in table 3.9 for the linear short term Phillips curve are mixed. On the one hand, the phase variable ω is forecasted with a better accuracy than possible with a VAR for both models. On the other hand, the results for λ of both Lotka-Volterra-like models are relatively poor compared to those of the VAR model, especially for the results of the CES production function. A conclusion is that if one wants to run prospective simulations for both state variables, the Goodwin-like models with a linear Phillips curve are less likely to provide accurate trajectories than those of a VAR. Table 3.10 reveals significant forecast improvements compared to the previous table. Results are quite similar for the Leontief case, whereas the performance for the CES case is significantly improved, especially in regards to the performance of λ. The combined performances

[23] A large evaluation period as been chosen in order to avoid any bias on the choice of the sample that can change the resulting outcome.

[24] Similar results are found after a sensitivity analysis with $\gamma = 2$.

[25] For the forecast, one can use the simulation made by taking the mean of hundreds of simulated paths of the stochastic differential model. For computational purposes, the trajectories are made from the deterministic counterpart of the model using a Runge-Kutta of order 4 as the discretization scheme.

	Leontief		CES	
	ω	λ	ω	λ
h = 1	**0.9410800**	1.271045	**0.9415192**	1.225308
h = 2	**0.8588958**	1.227962	**0.8634157**	1.134403
h = 3	**0.8038123**	1.246849	**0.8036210**	1.127487
h = 4	**0.7797888**	1.218206	**0.7845611**	1.088740
h = 5	**0.7861959**	1.221088	**0.7891970**	1.071780
h = 6	**0.8065483**	1.232296	**0.8050631**	1.069787
h = 7	**0.8022505**	1.273716	**0.7921797**	1.098498
h = 8	**0.8165254**	1.303057	**0.8029252**	1.112314

TABLE 3.10: Relative performance of the model against a VAR with the short term Phillips curve (3.7).

of the means of ω and λ indicate that the Goodwin model with the Phillips curve (3.7) outperforms the VAR model, for horizon $h = 2, \ldots, 8$. Table 3.11 shows that despite

	Leontief		CES	
	ω	λ	ω	λ
h = 1	**0.9409613**	1.279209	**0.9417142**	1.239200
h = 2	**0.8584490**	1.236147	**0.8630748**	1.156328
h = 3	**0.7935959**	1.258141	**0.7972700**	1.159880
h = 4	**0.7728046**	1.232525	**0.7834216**	1.118988
h = 5	**0.7709169**	1.237757	**0.7808093**	1.112729
h = 6	**0.7840967**	1.251712	**0.7939528**	1.113791
h = 7	**0.7694692**	1.297180	**0.7745961**	1.145692
h = 8	**0.7883737**	1.329552	**0.7895016**	1.163751

TABLE 3.11: Relative performance of the model against a VAR with the short term Phillips curve (3.8).

some improvement for ω, the global performance over the eight periods is slightly below that of the global performance of the short term Phillips curve displayed in Table 3.10.

The intermediate conclusion is twofold: (i) the Leontief production function with the short term Phillips curve (3.8) leads to better forecasting results for the wage share; (ii) generally, the model that outperforms the others is the model with a CES production function with a short term Phillips curve (3.7).

3.5.3 Results for models using a rolling window estimation

Results of the previous exercise is made using the whole sample, however, as previously discussed, estimated parameters may be biased due to the structural breaks.[26] Therefore, for the sake of comparison, I perform the same exercise by estimating the model on a rolling window. In using this technique, the length of the dataset remains throughout

[26]This observation holds for the Lotka-Volterra-like model as well as the VAR.

each estimation. Meaning that, the methodology presented *supra* does not hold up anymore because at each iteration of the loop, the dataset size was strictly increasing. In this exercise, I chose a sample size of 150 (or 37.5 years),[27] this window size will stay fixed along the backtesting, meaning that as soon as a new forward looking data point is taken, the first point of the preceded window is dropped.

Tables 3.12, 3.13, and 3.14 display the new performance ratio with the explained methodology. Results are globally similar than the previous attempt with a growing sample. Nonetheless, with respect to Section 3.5.2, while ratio of the error forecasts of the state variable for the employment rate λ show an upper ratio until horizon 3, it shows significant improvement on the remaining horizon. In addition, although the results seem to be lower for the wage share ω in all the cases compared with the backtesting counterpart showed by Tables 3.9, 3.10, and 3.11 when structural breaks are included, it is worth mentioning that the baseline model–the VAR–also changes and is improved by the break.

	Leontief		CES	
	ω	λ	ω	λ
h = 1	**0.9887657**	1.299865	**0.9845405**	1.255403
h = 2	**0.9771525**	1.291017	**0.9688992**	1.212261
h = 3	**0.9253679**	1.299273	**0.9125817**	1.202400
h = 4	**0.9218033**	1.229424	**0.9126602**	1.123260
h = 5	**0.9453638**	1.187798	**0.9342223**	1.073091
h = 6	**0.9632842**	1.179657	**0.9574811**	1.037484
h = 7	**0.9595548**	1.167956	**0.9482671**	1.020718
h = 8	**0.9537263**	1.170670	**0.9436899**	1.012033

TABLE 3.12: Relative performance of the model with break against a VAR with the short term Phillips curve (3.6).

	Leontief		CES	
	ω	λ	ω	λ
h = 1	**0.9902910**	1.296373	**0.9854354**	1.256501
h = 2	**0.9735293**	1.287830	**0.9653849**	1.215953
h = 3	**0.9146372**	1.296698	**0.9039285**	1.202727
h = 4	**0.9233361**	1.230511	**0.9099855**	1.126275
h = 5	**0.9431289**	1.188639	**0.9278705**	1.074053
h = 6	**0.9567395**	1.180956	**0.9479620**	1.042599
h = 7	**0.9553689**	1.168568	**0.9407818**	1.020840
h = 8	**0.9741219**	1.170632	**0.9517403**	1.011379

TABLE 3.13: Relative performance of the model with break against a VAR with the short term Phillips curve (3.7).

[27]This frame 150 represents slightly more that half of the sample.

	Leontief		CES	
	ω	λ	ω	λ
h = 1	**0.9889621**	1.297678	**0.9846496**	1.260712
h = 2	**0.9689416**	1.290015	**0.9618207**	1.222345
h = 3	**0.9043663**	1.300883	**0.8968094**	1.212087
h = 4	**0.9187525**	1.234588	**0.9060300**	1.138082
h = 5	**0.9369310**	1.193604	**0.9219436**	1.085778
h = 6	**0.9530819**	1.187343	**0.9423119**	1.056996
h = 7	**0.9579636**	1.176195	**0.9349791**	1.035789
h = 8	**0.9835237**	1.178574	**0.9498646**	1.026396

TABLE 3.14: Relative performance of the model with break against a VAR with the short term Phillips curve (3.8).

3.5.4 Understanding the Results for the Employment Rate

Throughout the backtesting, the employment rate, λ, did not perform as well as the other state variable ω. To understand why, one should focus on the performance and appreciate how the sub-periods have influenced the determination of the global measures. For the sake of clarity, one can consider the time horizon $h = 4$, in the column CES, and λ of table 3.11, 1.118988. This value is D_4^x/D_4^{VAR}, where x stands for the Goodwin model with a CES production function and a short term Phillips curve defined by (3.8). The goal of the proposed exercise is to plot all components that made D_4^x and D_4^{VAR}, namely, $\forall i_4 = 1, \ldots, 100$, $d_\gamma(x_{i_4}^x, x_{i_4}^t)$, and $d_\gamma(x_{i_4}^{VAR}, x_{i_4}^t)$.

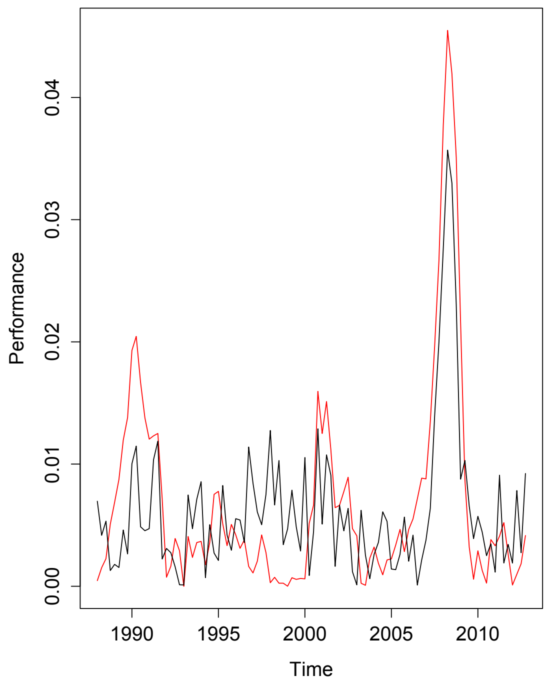

FIGURE 3.6: The performance of λ for the Goodwin model, in red, and for the VAR model, in black.

Figure 3.6 shows the timeseries $d_\gamma(x_{i_4}^{VAR}, x_{i_4}^t)$ in black and the timeseries $d_\gamma(x_{i_4}^x, x_{i_4}^t)$ in red. At any given date, when the red line is above the black line, it means that the error made by the VAR for a four-period-ahead forecast is lower than the error of the model x. Qualitatively, one sees that the VAR outperforms the model x for the three last recession periods: (i) the Kuwait invasion (reported from 1990:Q3 to 1991:Q1.[28]); (ii) the burst of the dot-com bubble (reported from 2001:Q1 to 2001:Q4); and (iii) the subprime mortgage crisis (reported from 2007:Q4 to 2009:Q2). Removing those periods, the global performance for every horizon will turn out to be

h=1	h=2	h=3	h=4	h=5	h=6	h=7	h=8
0.9761	**0.980**	1.003	**0.9733**	**0.9700**	**0.9647**	**0.9839**	**0.9973**

TABLE 3.15: Performance of λ without crises periods.

An empirical extension with the goal of increasing the reliability of the prediction made for λ would be the theoretical extension proposed by Keen (1995) and is a promising avenue for future research. Keen (1995) introduced debt into the dynamic, and by doing so, necessarily introduced the investment function that plays an important role in the global dynamics, especially for the dynamic of λ,

$$\dot{\lambda} = \lambda \left(\frac{\kappa(\pi)}{\nu} - [\alpha + \beta + \delta] \right),$$

where $\pi = 1 - \omega - rd$, with r as interest rate and d representing debt-to-GDP ratio. If one sets $\kappa(x) = x$ and $r = 0$, the equation for $\dot{\lambda}$ is the same as in system 3.1. By breaking the equality between profit and investment, Keen had a new dimension with debt. Although his dissipative model is three-dimensional, it gives room for emerging phenomenon such the Minsky moment and financial instability. Nonetheless, Keen's assumption allows for more flexibility on the global behavior of λ that may better capture the crisis effect. Note that van der Ploeg mentioned this extension as an extension within the Goodwin framework (see. Van der Ploeg (1985), p.229).

3.6 Concluding Remarks and Further Extensions

This paper provided a global methodology to assess a class of macroeconomic models such as Goodwin's. I proposed a methodology to estimate continuous-time macroeconomic models with low-frequency data. An experiment was carried out by testing the Goodwin model and one of its extensions, the Van der Ploeg (1985) model. To date, the results regarding the empirical success of the Goodwin model have been mixed. This

[28]The recession dates reported are from the NBER.

paper tackled the question of the empirical estimation Goodwin model from a different perspective. Instead of inferring the model parameter by parameter or equation by equation, the author's approach allowed for an assessment of the model as a whole.

Results of the estimation show that the cycles have structurally changed along the decades, meaning that, for the US, three sub-periods can be drawn out from the data: (i) 1948:Q1-1984:Q1; (ii) 1984:Q2-2000:Q1; and (iii) 2000:Q2-2015:Q4. For each of the sub-periods, the economy endowed with a CES technology outperforms the Leontief counterpart in explaining the data's behavior. It has been shown that a nonlinear function for the short term Phillips curve describes better the data than a linear one. However some improvements can be made in finding a more suitable nonlinearity for the short term Phillips curve, especially for the last sub-period. Another improvement would be to add a lagged relation to the Phillips curve, first, to improved the reliability of the short term Phillips curve and, second, to fit better the data. This last extension would involve delayed differential equations' theory.

Additionally, a backtesting strategy based on a out-of-sample forecast was considered. The aim was to assess whether the model may be used for prospective scenarios. To do so, I compared its forecast ability against a VAR and obtained a globally positive result. That is to say that a global performance measure showed that few of the Goodwin-like models tested in the paper were able to provide satisfactory results up to a medium run horizon, i.e. two years.

This paper added to a growing body of work that has developed theoretical models all based on the Goodwin-Lotka-Volterra model. The methodology used in the paper may been seen as a starting point for further empirical studies of extensions of Goodwin based models. A desirable extension would involve testing the Keen model and evaluating the investment function in order to provide a more accurate forecast for λ and, in the meantime, a more precise estimate of the depreciation rate of capital. Furthermore, the estimation framework can be improved by extending the methodology to missing data points in order to allow the estimation technique to cope with various frequencies within the dataset.

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Chapter 4

Minskyan Classical Growth Cycles: Stability Analysis of a Stock-Flow Consistent Macrodynamic Model

Abstract: This paper follows Van der Ploeg (1987)'s research program in testing both its extension of Goodwin (1967)'s predator-prey model and the Minsky Financial Instability Hypothesis (FIH) proposed by Keen (1995). By endowing the production sector with CES technology rather than a Leontief, van der Ploeg showed that the possible substitution between capital and labor transforms the close orbit into a stable focus. Furthermore, Keen (1995)'s model relaxed the assumption that profit is equal to investment by introducing a nonlinear investment function. His aim was to incorporate Minsky's insights concerning the role of debt finance. The primary goal of this paper is to incorporate additional properties, inspired by van der Ploeg's framework, into Keen's model. Additionally, we outline possibilities for production technology that could be considered within this research program. Using numerical techniques, we show that our new model keeps the desirable properties of Keen's model. However, we also demonstrate that when the economy is endowed with a class of production function that includes the Cobb-Douglas and the linear technology as limit cases, the unique stable equilibrium is an economically desirable one. We conclude that CES production function is a more suitable assumption for empirical purposes than the Leontief counterpart. Finally, we show, using numerical simulations, that under plausible calibration, the model endowed with a CES production function eventually lose the cyclical property of Goodwin's model.

JEL Codes:: C02, E10, E22, G01.

Keywords: Prey-predator; Goodwin model; Keen model; Minsky's financial instability hypothesis;Dynamical systems.

4.1 Introduction

It has been almost half a century since Goodwin (1967) developed a model of endogenous real growth cycles. Based on a simple and well known dynamic–the nonlinear Lotka-Volterra prey-predator model–Goodwin's model appeals in its simplicity and can be easily applied by researchers in a variety of fields (physics, biology, etc.). Goodwin's growth cycle is a simple dynamic model of distributional shares of output[1] and of (un)employment. In his model, he shows how accumulation takes the form of a cycle. The solution of the model describes a family of closed cycles in the state variables (workers' share and employment).

In the 1980s, van der Ploeg merged what he called "the neo-Keynesian" and the "neoclassical views." He extended the Goodwin-Lotka-Volterra model by incorporating constant elasticity of substitution, or CES production technology, thus capital and labor are no longer complementary factors of production. By doing so, he relaxed the original assumption of a constant capital-to-output ratio made by Goodwin. The advantage of this relaxation is that both the Leontief technology underlying Goodwin's model and the more general technology underlying Solow (1956)'s model are incorporated as special cases. Furthermore, van der Ploeg showed that the choice of incorporating CES technology destroys the conservative nature of the Goodwin system, i.e. transforms a closed orbit in a stable focus. Therefore, perpetual cycles are replaced by damped cycles or monotonic convergence to the balanced growth equilibrium. The primary economic rationale of van der Ploeg's extension would be that improved profit will stimulate investment and thus increase output. This gain in output would in turn stimulate jobs and push wages up by wage negotiation. Through internal financing, firms would substitute labor by capital by firing and installing new capital. For example, in Goodwin's model, a higher wage share results in lower profits. The latter will negatively impact output growth and employment due to lower investment. This lower investment will eventually lead to a new boom in the "class struggle" cycle, which in turn will lead to a decrease in wages and increase in profits. In van der Ploeg's model, when substitution between factors of production is considered, a given period of high employment rate–the apex of

[1]In the sense of the GDP at factor cost, where the income approach of the GDP is summarized as the distribution of wages and profits.

the cycle–will induce a substitution of labor by capital rather than an increase in wages. Such substitution will allow for the dampening of employment and wage variations.

In van der Ploeg's framework, the equality between investment and profits always holds. In this paper, we propose to relax the equality by incorporating a Minskyan framework including debt. Minsky's Financial Instability Hypothesis (hereafter: FIH) links the expansion of credit with the rise of asset prices and the inherent fragility of the financial system. Although Minsky made his points clear, aided by convincing diagrams and incisive exploration of data, he refrained from presenting his ideas as a mathematical model. This task was taken up by Keen (1995), where a system of differential equations was proposed as a simplified model incorporating the basic features of Minsky's hypothesis. By adding a new dimension to Goodwin's model, Keen's dynamic model changes a conservative system into a dissipative one in which the dynamics display sensitive dependence on initial conditions. Specifically, this new model has a stable equilibrium defined in terms of the employment rate, the profit rate, and the debt-to-output ratio. Additionally, the system will converge to this equilibrium if the initial conditions are sufficiently close to what we will call the *good* equilibrium. However, for other initial conditions, the model bifurcates (see Pomeau and Manneville (1980)) and undergoes an unstable cyclical breakdown towards what we will call the *bad* equilibrium.

This paper shows that when the economy is endowed with CES production function, the properties of Keen's model are preserved. Furthermore, under the assumption of a class of production function being bounded by the Cobb-Douglas and the linear, the equilibrium that led to the collapse (i.e., the *bad* equilibrium) is no longer locally stable. We also show that the basin of attraction towards the *good* equilibrium is substantially larger when substitution is allowed. Finally, we indicate that the model with CES production function would be a more suitable candidate for estimation purposes than the Leontief counterpart.

This paper is organized as follows: Section 2 outlines the model that departs from Keen (1995) by incorporating CES production technology. Section 3 introduces the equilbria and the study of their local stability. Section 4 presents different properties deduced from the study of the basins of attraction. Finally, Section 5 offers concluding remarks and suggestions for further works.

4.2 The Model

Keen (1995) assumed a Leontief production technology in which the inputs of production–capital and labor–are not substitutable. The production function is defined as follows[2]

$$Y^k := \min\left(\frac{K^k}{\nu}, aL^k\right),$$

where Y^k is the real output, K^k the stock of productive capital, L^k the labor force and $a > 0$ the labor productivity. In this framework, the capital-to-output parameter $\nu > 0$ is constant and technology is assumed to be used at its maximal capacity. Van der Ploeg (1985) relaxed the assumption that capital and labor cannot be substituted with each other by endowing the economy with a CES production function

$$Y = C\left[bK^{-\eta} + (1-b)(e^{a_l t}L)^{-\eta}\right]^{-\frac{1}{\eta}}, \tag{4.1}$$

where $C > 0$ is the factor productivity (or efficiency parameter) and $b \in (0;1)$ that reflects the capital intensity of the production. The short-run elasticity of substitution between capital and labor is given by $\sigma_\eta := \frac{1}{1+\eta}$. The labor productivity is driven by a constant rate of growth a_l. It is worth recalling that the CES production function allows for three limit cases: (i) when $\eta \to +\infty$, one retrieves the Leontief production function; (ii) $\eta \to 0$ leads to the Cobb-Douglas production; (iii) if $\eta \to -1$ one recovers the linear production function.[3]

Real wages are set according to a short-run Phillips curve and we assume that the production sector behaves like a large oligopoly which adjusts the quantity produced so as to maximize its profit.[4] The first-order condition that characterizes the profit maximizing behavior implies that real wages equal their marginal return

$$\frac{\partial Y}{\partial L} = \mathrm{w}.$$

Thus, van der Ploeg broke Goodwin's assumption of a constant capital-to-output ratio. Therefore, it is now time-varying and is defined by

$$\nu_\eta(t) \quad := \quad \frac{K(t)}{Y(t)} = \frac{1}{C}\left(\frac{1-\omega(t)}{b}\right)^{-\frac{1}{\eta}},$$

[2]The superscript k stands for Keen.

[3]We confine ourselves to a real economy, so that the consumption price is normalized to 1.

[4]This minimal rationality argument is analogous to the assumption in Goodwin's model that the allocation of capital and labor is always at the diagonal of the (K^k, L^k)-plan, so that we have not only $Y^k = \min\left(\frac{K^k}{\nu}, aL^k\right)$ but also $Y^k = \frac{K^k}{\nu} = aL^k$.

where ω is the wage-to-output ratio. We endow ν_η with a subscript η since the dynamic will strongly depend upon η's value. Whenever $\eta \to +\infty$, the capital-to-GDP ratio is constant as in the Leontief case. The full model derivation of the Minskyan classical growth cycle model is available in Appendix C.1. It boils down to a three-dimensional system

$$\begin{cases} \dot{\omega} & = \omega \left[\left(\frac{\eta}{1+\eta} \right) [\Phi(\lambda) - a_l] \right] \\ \dot{\lambda} & = \lambda \left[\kappa(\pi) C \left(\frac{1-\omega}{b} \right)^{1/\eta} - \delta - a_l - \beta - \frac{1}{\eta(1-\omega)} \frac{\dot{\omega}}{\omega} \right] \\ \dot{d} & = d \left[r - \kappa(\pi) C \left(\frac{1-\omega}{b} \right)^{1/\eta} + \delta + \frac{\dot{\omega}}{(1-\omega)\eta} \right] + \kappa(\pi) - (1-\omega) \end{cases} \tag{4.2}$$

where $\Phi(.)$ is the short-term Phillips curve that depends positively upon λ, the employment rate. The second aggregate behavior is given by the function $\kappa(.)$, that controls the investment-to-output ratio and depends upon the profit share $\pi := 1 - \omega - rd$. In the latter, d stands for the debt-to-output ratio and $r > 0$ for the constant short term interest rate. Finally, the parameter $\delta > 0$ refers to the depreciation rate of capital and β to the rate of growth of the population.

In system 4.2, whenever $\eta \to +\infty$ and $C = 1/\bar{\nu}$, Keen (1995)'s model is retrieved. Thus, we recover a Leontief production function and, as previously stated, a constant capital-output ratio.

The growth rate of the economy is given by[5]

$$g := \frac{\dot{Y}}{Y} = \kappa(\pi) C \left(\frac{1-\omega}{b} \right)^{1/\eta} - \delta - \frac{\dot{\omega}}{(1-\omega)\eta}.$$

Here, as in both the van der Ploeg and Keen models, the behavior of households is fully accommodating in the sense that, given investment $I := \kappa(\pi)Y$, consumption is determined by the macro balance

$$C := Y - I = (1 - \kappa(\pi))Y$$

precluding a more general specification of household saving propensity.[6] Table 5.1 makes the stock-flow consistency of the model explicit. For this paper, we adopt the convention that

$$D := L - M^f,$$

where the net borrowing D is the difference between firm loans, L, and firms deposits, M^f. Furthermore, the model 4.2 is retrieved from Table 5.1 by assuming $r = r^M = r^L$.

[5]See Appendix C.1 for the computation.

[6]Studying the consequences of dropping Say's law will be the task of a forthcoming paper.

	Households	Firms		Banks	Sum
Balance Sheet					
Capital stock		K			K
Deposits	M^h	M^f		$-M$	
Loans		$-L$		L	
Sum (net worth)	X^h	X^f		X^b	X
Transactions		current	capital		
Consumption	$-C$	C			
Investment		I	$-I$		
Accounting memo [GDP]		$[Y]$			
Wages	W	$-W$			
Interests on deposits	$r^M M^h$	$r^M M^f$		$-r^M M$	
Interests on loans		$-r^L L$		$r^L L$	
Financial Balances	S^h	Π	-I	S^b	
Flow of funds					
Gross Fixed Capital Formation		I			I
Change in Deposits	$\dot{M}^h$	$-\dot{M}^f$		$-\dot{M}$	
Change in loans		$-\dot{L}$		$\dot{L}$	
Column sum	S^h	Π		S^b	I
Change in net worth	$\dot{X}^h = S^h$	$\dot{X}^f = \Pi + -\delta K$		$\dot{X}^b = \Pi^b$	$\dot{X}$

TABLE 4.1: Balance sheet, transactions, and flow of funds in the economy

With this set-up, we can readily verify that the accounting identity that requires investment to be equal to savings always holds. Furthermore, according to Nguyen-Huu and Pottier (2016), the channel of debt financing is not fully determined by the model. It does not distinguish between loanable funds and endogenous money creation since both rationales induce the same set of equations. For the numerical simulations that

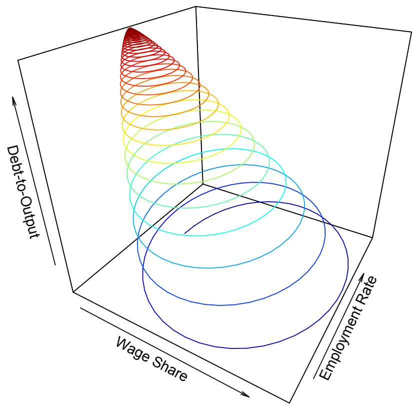

FIGURE 4.1: Keen's model with a Leontief production function and initial condition in the neighborhood of the *good* equilibrium.

follow, most of the parameters are borrowed from Keen (2013) and are explained below.

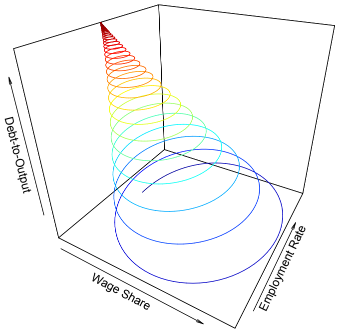

FIGURE 4.2: CES model with $\eta = 500$ and initial conditions in the neighborhood of the *good* equilibrium.

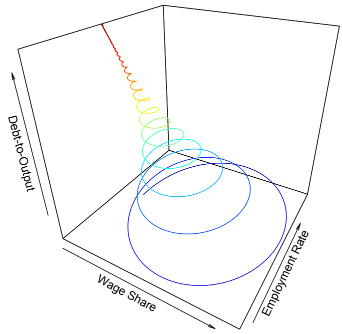

FIGURE 4.3: CES model with $\eta = 100$ and initial conditions in the neighborhood of the *good* equilibrium.

Figure 4.1 shows the (ω, λ, d)-space, or the phase space, of Keen's model converging towards the *good* equilibrium. Qualitatively, we observe that Goodwin's cyclical behavior is dampened when the trajectory approaches its equilibrium value. Remember that, in the CES setting, Figure 4.1 refers to the limit case

$$\lim_{\eta \to +\infty} \nu_\eta(t) = \nu,$$

of a constant capital-to-output ratio.

Turning to the time-varying behavior of the capital-to-output ratio, Figures 4.2 and 4.3 show the counterpart model where the elasticity of substitution for capital and labor are $\sigma_{500} = \frac{1}{1+500} \approx 0.2\%$ and $\sigma_{100} = \frac{1}{1+100} \approx 1\%$, respectively. By allowing for substitution between capital and labor (while maintaining other parameters and beginning again near the *good* equilibrium), we observe that the cycles are more muted when η is lower. This characteristic echoes the stable focus behavior demonstrated by van der Ploeg as opposed to the closed orbit showed by Goodwin's dynamic. A primary rationale to

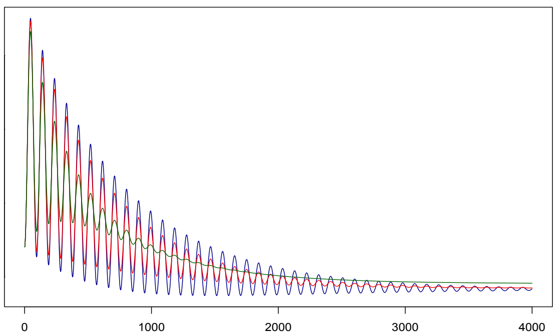

FIGURE 4.4: Evolution of ω in the previous simulations. In blue: $\eta = +\infty$. In red: $\eta = 500$. In green: $\eta = 100$.

explain the difference might be that a gain in profit boosts investment and output. Thus, employment will increase and so will total wages. Indeed, when $\eta > 0$, the wage share, ω, increases according to the wage negotiation curve

$$\frac{\dot{\omega}}{\omega} = \frac{\eta}{1+\eta}\left(\Phi(\lambda) - \alpha\right).$$

This rise in ω will increase the capital-to-output ratio ν_η since

$$\frac{\dot{\nu}_\eta}{\nu_\eta} = \frac{\dot{\omega}}{(1-\omega)\eta}.$$

It follows that firms will tend to substitute labor by capital by firing and installing new capital, because the growth of the capital-output ratio enters negatively in the evolution of the employment rate (see Appendix C.1).

Figure 4.4 shows the timeseries for the wage share ω as obtained by the three different settings used for the production function. These simulations clarify the difference in the evolution of the wage share and its cyclical properties (especially the amplitude of the cycles). The blue curve represents the Leontief technology. Interestingly, it demonstrates a more volatile behavior than its counterparts: (i) the CES with $\eta = 500$ (in red); and (ii) the CES with $\eta = 100$ (in green). We qualitatively observe that the amplitude of cycles decreases as η increases, while their periodicity remains similar. In other words, the qualitative change induced by the allowance of the capital-labor substitution affects only the amplitude of cycles, remaining unchanged the long-run trend.[7] Finally, Figure 4.5 illustrates the behavior of the system in a case with low substitution ($\eta = 100$), with the usual parameters (see Appendix C.2), except for $\delta = 0.02$.

4.3 Equilibria

All equilibria can be found by an exhaustive examination of three cases. First, $\omega_1 \neq 0$, meaning that the wage share is not zero. This point will be labeled as the *good* equilibrium. Second, $\omega_2 = 0$, $\lambda_2 \neq 0$, which provides an economically meaningless equilibrium. Finally, $\omega_3 = 0$, $\lambda_3 = 0$, which provides two equilibria: an *obvious* and a *bad* equilibrium.

4.3.1 The *Good* Equilibrium

Prior to the derivation of this equilibrium, it is worth mentioning that the point $(\omega_1, \lambda_1, d_1)$ refers to the *good* equilibrium. Using the equilibrium condition of ω_1, we find that

[7] In addition, it is worth mentioning that the equilibrium points differ slightly depending on the value of η.

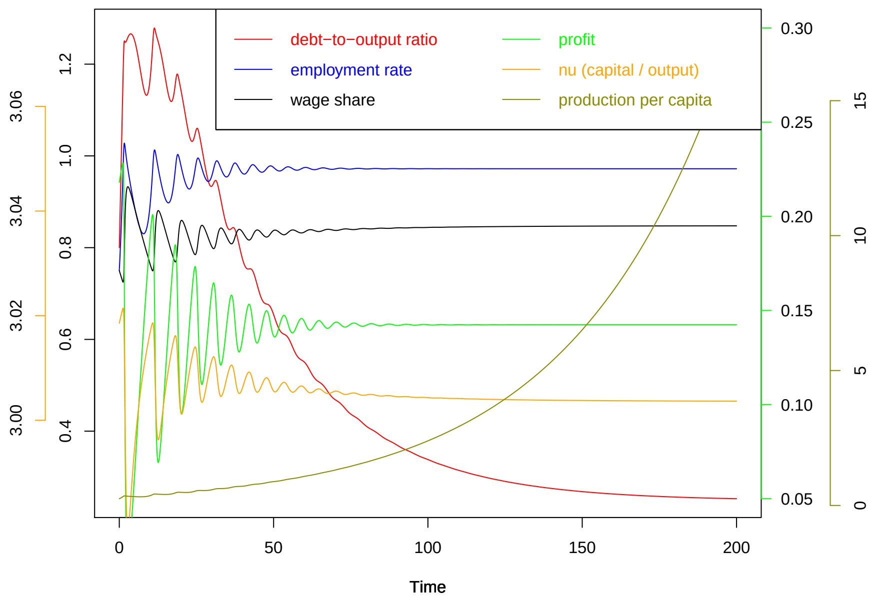

FIGURE 4.5: Illustration of the system dynamic with benchmark parameters and $\eta = 100$.

$\lambda_1 = \Phi^{-1}(a_l)$. Next, using the equilibrium condition of λ_1, $\kappa(.)$ can be written in terms of ω_1 so that

$$\kappa(\pi_1) = \frac{a_l + \beta + \delta}{C}\left(\frac{1-\omega_1}{b}\right)^{-\frac{1}{\eta}} =: \zeta_1(\omega_1). \tag{4.3}$$

Plugging equation (4.3) into the equilibrium condition of d_1 yields d_1 as a function of ω_1:

$$d_1(\omega_1) = \frac{1-\omega_1-\zeta_1(\omega_1)}{r - a_l - \beta}. \tag{4.4}$$

Plugging equation (4.4) into the $\kappa(.)$ of equation (4.3) leads to a nonlinear equation that ω_1 should satisfy. Depending on the specification chosen for κ,[8] we solve this equation numerically so that we get ω_1. Next, we find d_1 using equation (4.4). As with the Goodwin and Solow models, at the equilibrium point, the real growth rate of the economy is given by

$$g = \frac{\dot{Y}}{Y} = \kappa(\pi_1)C\left(\frac{1-\omega_1}{b}\right)^{1/\eta} - \delta = a_l + \beta.$$

One can note that, under reasonable specifications, this equilibrium exists. The right hand side of equation (4.3) is a function of ω_1 that equals 0 at $\omega_1 = 1$ and $\zeta_1(0) = \frac{a_l+\beta+\delta}{C}b^{\frac{1}{\eta}}$ at 0; while the left hand side equals κ_0 for $\omega_1 = 1$ and $\kappa\left(1 - \frac{r}{r-a_l-\beta}(1-\zeta_1(0))\right)$

[8]An example with an affine function is provided in Appendix C.4. It shows that, with the simplest case, a closed-form expression for the equilibrium could not be written.

for $\omega_1 = 0$. As both sides are continuous function of ω, it suffices that $\kappa\left(1 - \frac{r}{r-a_l-\beta}\left(1-\zeta_1(0)\right)\right) < \zeta_1(0)$ to insure the existence of the *good* equilibrium. For $\eta < 0$, $\zeta_1(0)$ takes high values (often above 1), so that the previous inequality holds. For $\eta > 0$, $1 - \frac{r}{r-a_l-\beta}\left(1-\zeta_1(0)\right)$ is often negative (it is negative as long as $\frac{a_l+\beta+\delta}{C} > \frac{1}{1-\zeta_1(0)}$), so that $\kappa_0 < \zeta_1(0)$ insures the existence of the equilibrium (because κ is increasing). As $\zeta_1(0) \in \left[0; \frac{a_l+\beta+\delta}{C}\right]$ in this case, assuming e.g. $\kappa(0) = 0$ is a sufficient condition for the existence of the *good* equilibrium.

In addition, one can show that, under reasonable specifications, ω_1 is positive. Indeed, as equation (4.3) rewrites $\omega_1 = 1 - b\left(\frac{C\zeta_1}{a_l+\beta+\delta}\right)^{-\eta}$, one just needs that $\left(\frac{C\zeta_1}{a_l+\beta+\delta}\right)^{\eta} > b$ to insure a non negative ω_1. Depending on the sign of η, this is equivalent to ζ_1 being above (resp. below) $\frac{a_l+\beta+\delta}{C} b^{\frac{1}{\eta}}$ for $\eta > 0$ (resp. $\eta < 0$). Hence, as long as the image of κ is entirely on the right side of this value, ω_1 is positive.

4.3.2 The *Slavery* Equilibrium

This second equilibrium is economically meaningless. It would suggest that wages are null while employment is still positive, and would be interpreted as characterizing *slavery*. Its derivation can be sketched in the same manner as before. The following function $\zeta_2(.)$ can be derived from system (4.2) so that

$$\kappa\left(1 - rd_2\right) Cb^{-\frac{1}{\eta}} - \delta = a_l + \beta + \frac{\Phi\left(\lambda_2\right) - a_l}{1+\eta} =: \zeta_2\left(\lambda_2\right). \tag{4.5}$$

Equation (4.5) together with system 4.2 gives

$$d_2\left(\lambda_2\right) = \frac{1 - \frac{b^{\frac{1}{\eta}}}{C}\left(\zeta_2\left(\lambda_2\right) + \delta\right)}{r - \zeta_2\left(\lambda_2\right)}. \tag{4.6}$$

Finally, plugging equation (4.6) into equation (4.5) gives a value satisfied by λ_2. Hence, d_2 can be deduced from equation (4.6). Note that whenever $\eta \to +\infty$ we retrieve Grasselli and Lima (2012) results.

4.3.3 The *Obvious* and The *Bad* Equilibria

This condition gives us two kinds of equilibria. On the one hand, *obvious* equilibria can be found by solving system 4.2, so that

$$\left(\bar{\omega}_3, \bar{\lambda}_3, \bar{d}_3\right) = \left(0, 0, \bar{d}_3\right)$$

where d_3 is any solution of

$$d\left[r-\kappa(1-rd)C\left(\frac{1}{b}\right)^{1/\eta}+\delta\right]+\kappa(1-rd)-1=0 \tag{4.7}$$

is an equilibrium point for (4.2). Note that this condition for finding the equilibrium is similar to Grasselli and Lima (2012) if we identify $C(1/b)^{-1/\eta}=\nu$. Therefore, we expect this equilibrium to be unstable in the same way that $(\bar{\omega}_3,\bar{\lambda}_3)$ is a saddle point for the Goodwin model.

On the other hand, another equilibrium with infinite debt can be found using the change of variable: $\frac{1}{d}=: u$. We get that

$$\begin{cases}\dot{\omega} &= \omega\left[\left(\frac{\eta}{1+\eta}\right)[\Phi(\lambda)-a_l]\right]\\ \dot{\lambda} &= \lambda\left[\kappa(\pi(u))C\left(\frac{1-\omega}{b}\right)^{1/\eta}-\delta-a_l-\beta-\frac{1}{(1-\omega)}\left(\frac{1}{1+\eta}\right)[\Phi(\lambda)-a_l]\right]\\ \dot{u} &= -u\left[r-\kappa(\pi(u))C\left(\frac{1-\omega}{b}\right)^{1/\eta}+\delta+\frac{\omega}{(1-\omega)}\left(\frac{1}{1+\eta}\right)[\Phi(\lambda)-a_l]\right]-u^2\left(\kappa(\pi(u))-(1-\omega)\right)\end{cases} \tag{4.8}$$

with $\pi(u):=1-\omega-r/u$. Therefore, the point $(\bar{\omega}_3,\bar{\lambda}_3,\bar{u}_3)=(0,0,0)$, or $(\bar{\omega}_3,\bar{\lambda}_3,\bar{d}_3)=(0,0,+\infty)$ is an equilibrium. It can be interpreted as representing the collapse of the economy induced by over-indebtedness, where employment and wage converge towards 0 and debt increases constantly towards infinity. This equilibrium is labeled as the *bad* equilibrium.

4.3.4 Local Stability Study

This subsection seeks to present the Jacobian matrices of the two models previously presented (both with and without the change of variable). This step will help us to analyze the local stability of the equilibria displayed above. Note that phenomenological functions $\Phi(.)$ and $\kappa(.)$ are similar to those of Keen (2013). For the sake of clarity, in what follows, we use $\kappa(\pi):=\kappa(1-\omega-rd)$. The Jacobian associated with system (4.2) is

$$J(\omega,\lambda,d)=\begin{bmatrix}\left(\frac{\eta}{1+\eta}\right)[\Phi(\lambda)-a_l] & \omega\left(\frac{\eta}{1+\eta}\right)\Phi'(\lambda) & 0\\ \epsilon_1 & \epsilon_2 & \lambda\left[-r\kappa'(\pi)C\left(\frac{1-\omega}{b}\right)^{1/\eta}\right]\\ \epsilon_3 & \frac{d}{1+\eta}\left(\frac{\omega}{1-\omega}\right)\Phi'(\lambda) & \epsilon_4\end{bmatrix}$$

with

$$\begin{aligned}
\epsilon_1 &= \lambda\left[-\kappa'(\pi)C\left(\frac{1-\omega}{b}\right)^{1/\eta}-\frac{1}{\eta}\kappa(\pi)\frac{C}{b^{1/\eta}}(1-\omega)^{1/\eta-1}-\right.\\
&\left.\frac{1}{(1-\omega)^2}\left(\frac{1}{1+\eta}\right)[\Phi(\lambda)-a_l]\right]\\
\epsilon_2 &= \kappa(\pi)C\left(\frac{1-\omega}{b}\right)^{1/\eta}-\delta-a_l-\beta-\frac{1}{(1-\omega)}\left(\frac{1}{1+\eta}\right)[\Phi(\lambda)-a_l]-\\
&\lambda\frac{1}{(1-\omega)}\frac{1}{1+\eta}\Phi'(\lambda)\\
\epsilon_3 &= d\left[\kappa'(\pi)C\left(\frac{1-\omega}{b}\right)^{1/\eta}+\frac{1}{\eta}\kappa(\pi)\frac{C}{b^{1/\eta}}(1-\omega)^{1/\eta-1}+\frac{1}{1+\eta}\left(\Phi(\lambda)-a_l\right)\frac{1}{(1-\omega)^2}\right]\\
&-\kappa'(\pi)+1\\
\epsilon_4 &= r-\kappa(\pi)C\left(\frac{1-\omega}{b}\right)^{1/\eta}+\delta+\frac{\omega}{(1-\omega)}\left(\frac{1}{1+\eta}\right)[\Phi(\lambda)-a_l]+rd\kappa'(\pi)C\left(\frac{1-\omega}{b}\right)^{1/\eta}\\
&-r\kappa'(\pi)
\end{aligned}$$

At the equilibrium point $(\bar{\omega}_0, \bar{\lambda}_0, \bar{d}_0) = (0, 0, \bar{d}_0)$, the Jacobian moves down to a lower triangular matrix

$$J(0,0,d_0) = \begin{bmatrix} \left(\frac{\eta}{1+\eta}\right)[\Phi(0)-a_l] & 0 & 0 \\ 0 & \epsilon_2' & 0 \\ \epsilon_3' & 0 & \epsilon_4' \end{bmatrix}$$

with

$$\begin{aligned}
\epsilon_2' &= \kappa(1-rd_0)C\left(\frac{1}{b}\right)^{1/\eta}-\delta-a_l-\beta-\left(\frac{1}{1+\eta}\right)[\Phi(0)-a_l])\\
\epsilon_3' &= d\left[\kappa'(1-rd_0)C\left(\frac{1}{b}\right)^{1/\eta}+\frac{1}{\eta}\kappa(1-rd_0)\frac{C}{b^{1/\eta}}+\frac{1}{1+\eta}\left(\Phi(0)-a_l\right)\right]-\kappa'(1-rd_0)+1\\
\epsilon_4' &= \left[r-\kappa(1-rd_0)C\left(\frac{1}{b}\right)^{1/\eta}+\delta\right]+rd_0\kappa'(1-rd_0)C\left(\frac{1}{b}\right)^{1/\eta}-r\kappa'(1-rd_0).
\end{aligned}$$

The Jacobian's real eigenvalues are given by diagonal entries and it is not easy to determine the sign of d_0. Nevertheless, we note that, for a sufficiently large value of $\pi_0 := 1 - rd_0$, ϵ_2' is positive, whereas a sufficiently small value of π_0 (i.e., a large enough value of d_0) makes ϵ_4' non-negative. We conclude that this equilibrium point is likely to be unstable.

Although Grasselli and Lima (2012) could analytically retrieve all equilibria, our model is too intricate to do so. Thus, we will later compute the eigenvalues of the Jacobian matrices corresponding to each equilibrium under reasonable calibration.

In order to analyze the local stability of the *bad* equilibrium point $(0, 0, +\infty)$, we denote $\kappa(\pi(u)) := \kappa(1 - \omega - r/u)$ and the corresponding Jacobian matrix of system (4.8) is

$$J(\omega, \lambda, u) = \begin{bmatrix} \left(\frac{\eta}{1+\eta}\right)[\Phi(\lambda) - a_l] & \omega\left(\frac{\eta}{1+\eta}\right)\Phi'(\lambda) & 0 \\ \epsilon_1 & \epsilon_2 & \lambda\left[\kappa'(\pi(u))C\left(\frac{1-\omega}{b}\right)^{1/\eta}\frac{r}{u^2}\right] \\ \epsilon_3'' & \frac{-u}{1+\eta}\left(\frac{\omega}{1-\omega}\right)\Phi'(\lambda) & \epsilon_4'' \end{bmatrix}$$

with

$$\begin{aligned} \epsilon_3'' &= -u\left[\kappa'(\pi(u))C\left(\frac{1-\omega}{b}\right)^{1/\eta} + \kappa(\pi(u))C\frac{(1-\omega)^{1/\eta-1}}{b^{1/\eta}}\frac{1}{\eta} + \right. \\ &\quad \left.\left(\frac{1}{1+\eta}(\Phi(\lambda) - a_l)\right)\frac{1}{(1-\omega)^2}\right] - u^2\left[-\kappa'(\pi(u)) + 1\right] \\ \epsilon_4'' &= -\left[r - \kappa(\pi(u))C\left(\frac{1-\omega}{b}\right)^{1/\eta} + \delta + \frac{\omega}{(1-\omega)}\left(\frac{1}{1+\eta}\right)[\Phi(\lambda) - a_l]\right] \\ &\quad + \frac{r}{u}\kappa'(\pi(u))C\left(\frac{1-\omega}{b}\right)^{1/\eta} - 2u\left[\kappa(\pi(u)) - (1-\omega)\right] - \kappa'(\pi(u))r \end{aligned}$$

Despite the previous comment on the equilibrium points, since $\bar{\omega} = 0$ and $\bar{\lambda} = 0$, the Jacobian matrix is diagonal at this point. Thus, its eigenvalues are summarized by the diagonal terms

$$\left(\frac{\eta}{1+\eta}\right)[\Phi(0) - a_l], \qquad \kappa_0 C\left(\frac{1}{b}\right)^{1/\eta} - (\delta + a_l + \beta) - \left(\frac{1}{1+\eta}(\Phi(0) - a_l)\right),$$

$$-(r - \kappa_0 C\left(\frac{1}{b}\right)^{1/\eta} + \delta).$$

The sign of the eigenvalues will depend on the parameter η, which is assumed to belong to the set $]-1; +\infty[$. Indeed, when assuming $\Phi(0) < 0$, where wages decrease below some positive employment rate threshold, the first eigenvalue has the opposite sign of η. Furthermore, the remaining eigenvalues are negative if and only if

$$\kappa_0 = \inf_{\pi \in \mathbb{R}}\{\kappa(\pi)\} < \frac{b^{\frac{1}{\eta}}}{C}\min\left(r + \delta\,,\; a_l + \delta + \beta + \frac{1}{1+\eta}\left(\Phi(0) - a_l\right)\right).$$

Finally, given that $\Phi(0) < 0$, the *bad* equilibrium is stable if and only if $\eta > 0$ and the previous condition are fulfilled. Thus, if the elasticity of substitution is too high, i.e. below that of a Cobb-Douglas (as in the linear case e.g.), the *bad* equilibrium is unstable.

It is worth mentioning that depending on η and due to the condition of $\kappa_0 = \inf_{\pi \in \mathbb{R}} \in \mathbb{R}$, boundary condition of phenomenological function $\Phi(.)$ will have a significant effect on the stability of the *bad* equilibrium. Indeed, by taking reasonable values for r and δ, if

$\Phi(0) < -(1+\eta)(a_l+\delta+\beta)+a_l$ then $\kappa_0 < 0$ if one wants to guarantee the local stability of the *bad* equilibrium.

To numerically study the properties of other equilibria, we use a baseline calibration that closely follows Keen (2013).[9] Our model differs from Keen (2013)'s in that we assume that the productive sector is endowed with CES technology. We remind the reader that the capital-to-output ratio is not constant and equals

$$\nu_\eta(t) \quad = \quad \frac{1}{C}\left(\frac{1-\omega(t)}{b}\right)^{-\frac{1}{\eta}}.$$

In building the capital stock dataset, database makers assume an initial capital-to-output stock of about 3 (see Mc Isaac (2016)). In order to retrieve similar results for the Leontief case, we will assume that $\nu_\eta(t) \to 3$ whenever $\eta \to +\infty$, which implies that $C = 1/3$. On the other hand, the ratio $(1-\omega(t))/b$ should oscillate around 1 since b represents the share of capital in the production function in equation (4.1). In order to find a reasonable value, we will assume that $b = 1-\omega*$, where $\omega*$ is the *good* equilibrium value of the Leontief model counterpart. Finally, the parameter η will be tested for different values of $\sigma = 1/(1+\eta)$. First, we will test $\sigma = 0$, i.e. the Leontief case, where $\eta \to +\infty$. Second, the case where $\sigma = 1$–the Cobb-Douglas case–i.e., $\eta \to 0$–will be investigated. Third, the intermediary case between the last two, where $\sigma = 1\%$ (or $\eta = 100$) will be displayed. Finally, we will test a case where $\eta = -1/2$ to illustrate the case $\eta \in]-1;0[$ where the *bad* equilibrium is no longer locally stable.

TABLE 4.2: The summary table of stability for all equilibria.

Value of η	*good*	*obvious*	*slavery*	*bad*
$+\infty$	o	x	x	o
100	o	x	x	o
0	o	x	x	o
$-1/2$	o	x	x	x

Table 4.2 provides a summary of the tables available in Appendix C.3. An "o" means that the equilibrium is stable whereas an "x" stands for local instability. As expected, the *good* is always stable while the *obvious* and the *slavery* need not. However, we show that under this setting, the *bad* equilibrium is locally stable except when $\eta < 0$.

Figure 4.6 provides an example of a trajectory that converges towards a *good* equilibrium. The simulation begins at a point slightly behind the convergent spiral, at the bottom of the figure. At the beginning of the simulation, the trajectory displays ample cycles with a debt-to-output ratio that increases over the time. In other words, it shows large

[9] See Appendix C.2 for the details.

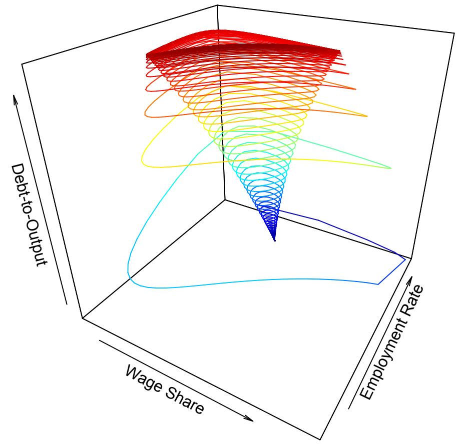

FIGURE 4.6: Simulation of a trajectory converging towards the *good* equilibrium with the calibration of Appendix C.2 and $\eta = 500$. The initial condition of the system is $(\omega, \lambda, d) = (0.80558764050, 0.9719722, 1.4995533)$, it is located at the bottom of the figure.

fluctuations of ω, the wage share, and λ, the employment rate. However, cycles will be less pronounced over time and after reaching a given level of debt-to-output, the productive sector begins deleveraging until it reaches its equilibrium point, at which of the state variable displays a finite value.

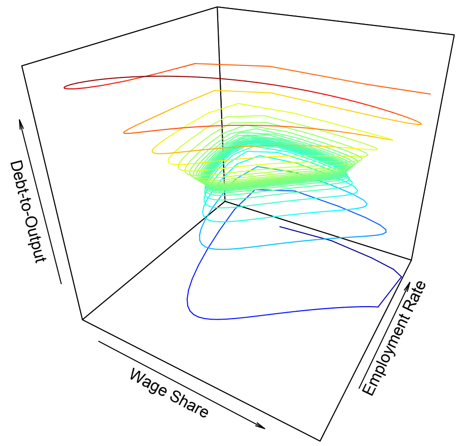

FIGURE 4.7: Simulation of a trajectory towards the *bad* equilibrium with the calibration of Appendix C.2 and $\eta = 500$. The initial condition of the system is $(\omega, \lambda, d) = (0.0.80558764045, 0.9719722, 1.4995533)$, which corresponds to the darkest blue point.

Figure 4.7 gives an example where the economy is no longer attracted by the *good* equilibrium. The only difference between Figures 4.6 and 4.7 is the initial condition of ω, which results in slightly higher profits. In the short to medium run, we observe a trajectory that is qualitatively similar to Figure 4.6. However, when the system arrives in a region where the economy is deleveraging, as in this example, the debt burden is too high and the productive sector can no longer reduce its debt. At this point, the economy's debt grows indefinitely.

4.4 Numerical Study of the Basins of Attraction

This section aims at quantitatively and qualitatively analyzing the specificity of each model presented above according to the respective basins of attraction. We consider four cases: (i) $\eta \to +\infty$; (ii) $\eta = 100$; (iii) $\eta = 0.5$;[10] and (iv) $\eta = -1/2$. Furthermore, we perform a sensitivity analysis of the following key parameters: a_l, the rate of labor productivity growth; β, the rate of population growth; δ, the depreciation rate of capital; and r the short term interest rate.

4.4.1 Methodology for the Basins of Attraction

We adopt a simple approach to evaluate the basins of attraction. We simulate our system taking initial conditions at each points of a grid in the (ω, λ, d) space (typically, this grid consists of 325 points in the cuboid $[0.5; 0.95] \times [0.6; 1] \times [0; 3]$). At the time horizon of $t = 200$, we compute the Euclidian distance of the set of simulated variables to their equilibrium values. We consider that a simulation converges towards the equilibrium whenever that distance falls below a chosen precision (0.5 in practice). Although our programs are flexible and allow us to evaluate many basins of attraction, depending on the choices of the equilibrium, the grid, the precision, the time horizon and the time-steps (for the Runge-Kutta fourth-order method used in the simulations), we stick to the specification described above and evaluate only the basin of attraction of the *good* equilibrium, the only one which is always stable. We finally plot the basins and compute their volume using Delaunay triangulation.

4.4.2 Main results

Figure 4.8 plots the basin of attraction of the *good* equilibrium in the Leontief case. In other words, every economy that is initialized in that set will numerically converge towards the *good* equilibrium. When the debt is low, the wage share must be high in order to keep the profit share at reasonable levels. In other words, if the wage share is not high enough, profit will be high and the investment share would skyrocket as a

[10]The case $\eta = 0.5$ is identified as being the closest to the Cobb-Douglas. However, as shown in Appendix C.6, when we derive the model with Cobb-Douglas production technology, we found that the wage share is no longer time-varying and equals, at all times, the output elasticity $1 - b$. Therefore, the Goodwin prey-predator logic does not hold anymore.

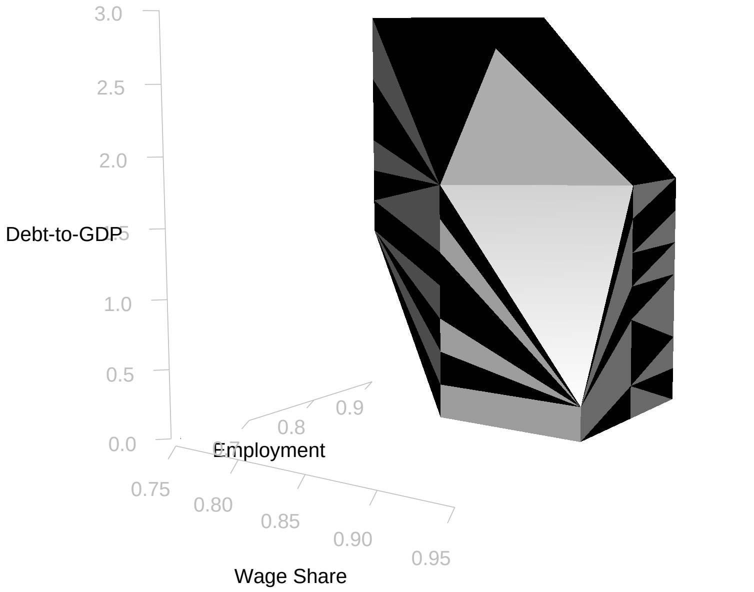

FIGURE 4.8: Basin of attraction of the *good* equilibrium for the Leontief production technology. The *good* equilibrium point is $(\omega, \lambda, d) = (0.865, 0.972, 1.50)$.

consequence. This would drive the economy to the Minskyan paradigm. In contrast, when the debt is high, the wage share should remain low. A consequence of low wage share would be higher profit, which provides a suitable situation for financial instability through the investment function since most of the value added would go to the debt services and no longer to workers and investment. A high wage share coupled with a high debt will prevent firms from deleveraging and, therefore reduce their debt.

Figure 4.9 shows the example of the basin of attraction of an economy endowed with a CES production function where $\eta = -0.5$ (between the Cobb-Douglas and the linear production function). In the numerical analysis of the set of points under consideration, nearly all of the state space is covered. However, the area where wage share, debt ratio, and employment levels are high does not lead to a convergence towards the *good* equilibrium. The latter case represents low profits and the incapacity of the corporate sector to deleverage, preventing the economy from converging towards the *good* equilibrium.

4.4.3 Sensitivity analysis

Table 4.3 shows the percentage of initial condition points that converge towards the *good* equilibrium. For instance, 79.1 represents the percentage of points within the considered area that converges under the benchmark calibration and $\eta = -0.5$. Given a higher

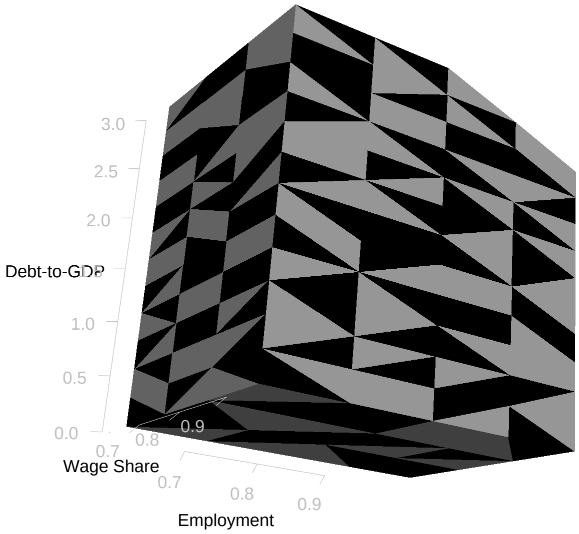

FIGURE 4.9: Basin of attraction of the *good* equilibrium for the CES production technology with $\eta = -0.5$. The *good* equilibrium point is $(\omega, \lambda, d) = (0.865, 0.972, 1.50)$.

TABLE 4.3: Parameter sensitivity of basins of attraction: percentage of points from the simulation grid falling into the basin of attraction of the *good* equilibrium. Values are reported in percentages.

parameters	benchmark:	a_l		r		δ		β	
η	(a_l, r, δ, β) $= (2, 4, 1, 1)$	− 1	+ 2.5	− 1	+ 5	− 0.5	+ 2	− 0	+ 3
$\eta = -0.5$	**96**	90	**100**	53	90	**96**	**94**	90	**100**
$\eta = 0.5$	93	**93**	93	**80**	**95**	93	93	**93**	93
$\eta = 100$	92	87	93	77	95	89	93	89	93
$\eta = 9^{15}$	26	17	28	17	28	21	34	14	53

elasticity of substitution $\sigma = 1/(1+\eta)$, the basin of attraction is usually larger. This is well illustrated by Figure 4.8, the basin of attraction of the Leontief case. Figure 4.9 plots the basin of attraction of the CES function with $\sigma = 2$. Note that CES production technology qualitatively shows a significantly larger basin of attraction than the Leontief one, even with very low substitution.

Bold values represent the largest number within each column. This value represents the calibration that shows the larger basin of attraction. Globally, the lower η, the larger the basin of attraction. However, for some calibrations, the basin of attraction of $\eta = 0.5$ is bigger than the one corresponding to $\eta = -0.5$. For the case $\eta = 0.5$, the only parameter that changes the value reported in Table 4.3 is the real interest rate r. For a higher value of r, the basin of attraction generally increases in volume, while for a lower value, it significantly decreases. The higher sensitivity of debt in the profits induced by the higher interest rate prevents, in almost all cases, the economy from falling into the FIH.

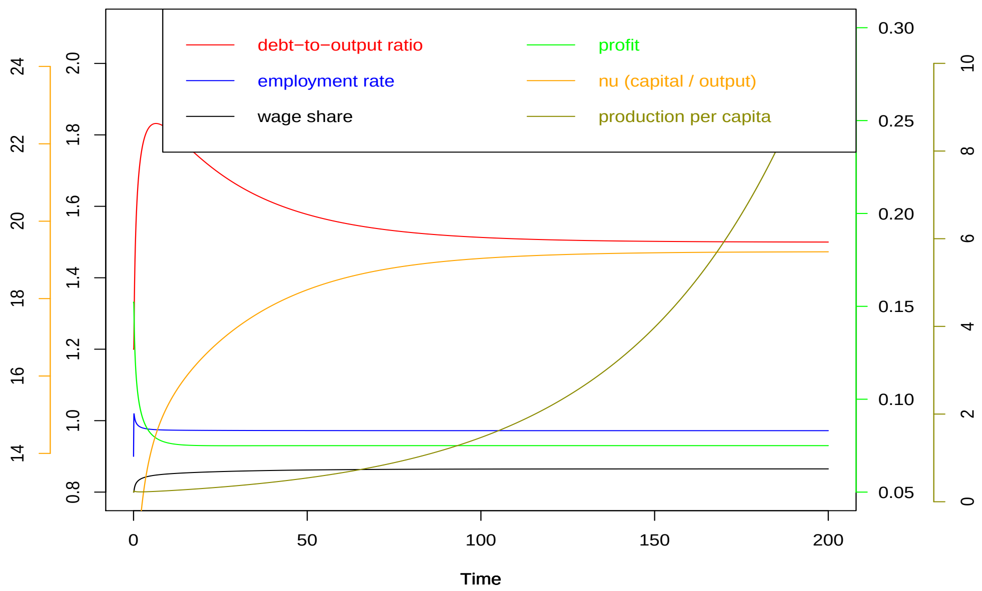

FIGURE 4.10: Illustration of the system dynamic with benchmark parameters except $b = 0.4$ and $\eta = 0.5$.

In system 4.2, an increase of the labor productivity growth rate, the depreciation rate of capital, or population growth negatively impact the employment rate dynamics. Thus, in the case with almost no substitution ($\eta = 9^{15}$), an increase in one of these parameters allows us to moderate the booms of the business cycles. This is similar to what substitution would induce (a decrease of the employment rate due to a lower relative cost of labor). This explains why in the Leontief case, Table 4.3 exhibits larger basins of attraction as a_l, δ and β increase. As soon as we allow for a limit degree of substitution, from almost 0% to $1/(100+1) \approx 1\%$, the surface of the basin of attraction substantially changes and highlights the the transformation recorded by difference of the Leontief to the CES case. This change is somewhat similar to van der Ploeg's results for the model's trajectory behaviors. However, these parameters only have a modest effect on the size of the basins of attraction when substitution is substantial. Indeed, Figure 4.10 plots the phase portrait of the state variable with the baseline calibration (see Appendix C.2) and a coefficient $\eta = 0.5$. In this simulation, the calibrated elasticity of substitution between capital and labor is $\frac{1}{1+0.5} \approx 66.66\%$, which is consistent with most of the empirical findings surveyed by Klump et al. (2012).[11] However, this simulation no longer shows the primary rationale of the Goodwin's endogenous economic fluctuations.

[11] Klump et al. (2012) surveyed a number of studies intended for developed counties in various time-frames (ca. 1800-2000). Almost 75% of the estimated elasticities showed a value between 0.5 and 1.

In other words, under what is ostensibly the most realistic calibration, our finding is that the model is no longer able to replicate the business cycles after the introduction of a reasonable substitution between factors of production. More research is needed to understand whether this is an inherent shortcoming of Goodwin-Keen inspired models or if the cyclical behavior could be recovered with some additional refinement.

4.5 Conclusions and Suggestions for Further Work

This paper presented multiple insights on how the class of CES production functions may significantly alter the dynamical landscape of Keen's model. The moderating effect induced by the flexibility of the use of factors of production on the dynamics of the system is a primary rationale of this paper's model. Substitution allows to mitigate the influence of wages on employment rate (and vice versa), so that both adjust more rapidly towards their equilibrium values.

Interestingly, numerical simulations showed that van der Ploeg's extension plugged into Keen's model increases the set of points that converges towards the *good* equilibrium. In other words, given the same behavioral conditions, the volume of the basin of attraction of the economically desirable equilibrium is greater for economies where some substitution can take place. Therefore, an intermediate conclusion would be that substitution between factors of production leads to the convergence of any reasonable initial condition towards the *good* equilibrium. Provided that substitution is strong enough (*i.e.* $\eta < 0$), it even ensures that the *bad* equilibrium is no longer stable.

Furthermore, using numerical techniques, we showed that when substitution is allowed, the change in the volume of the basin of attraction is globally less sensitive to a change of parameters. Consequently, a model with a CES function would be more suitable as a model for estimation. Indeed, when using statistical methods, the exact inference of a parameter is strongly unlikely. Thus, an economy endowed with CES production shows more robust properties to cope with numerical errors of the type inherent to the statistical analysis or flaws in the data. Additionally, realistic calibration of the model showed monotonic convergence rather than dampened cycles. Since the emergence of business cycles is an empirical fact (at least for some advanced economies), this inability to replicate business cycles under some calibration either suggests that the Keen model relies on complementary factors or that the phenomenological functions are not well calibrated.

This paper contributed to a growing body of work that has developed theoretical models based on the Goodwin-Lotka-Volterra model. The methodology used in this paper may be seen as a starting point for further analysis. We have tested the sensibility of the model to key parameters; another extension would be to test the robustness of the results with respect to changes in the behavioral functions. These changes would presumably affect the vector field of the system more substantially. Another natural extension might include testing the properties of a CES production function with three inputs (for instance by adding energy as in Acurio-Vásconez (2015)), two of which can be substituted with each other, while the third remains complementary. This would shed light on possibilities for the recovery of endogenous cycles in the model.

A different sort of application would involve dropping Say's law and modeling the behavior of the demand side. This would require either the introduction of the utilization rate of capital or an allowance for inventories. Such an application has been made by Grasselli and Nguyen-Huu (2016), who showed that the economy does not converge towards an equilibrium and exhibit limit cycles. This extension may be a way to reconcile the CES technology with the endogenous cycle theory within the Goodwin framework. Furthermore, another way to capture cycles in case of high sustainability between factors would be introduce money with a Taylor rule. This extension may be able to cope with monetary cycles that in turn could influence the real economy through that pattern.

In sum, we have seen that the basin of attraction changes substantially when substitution is allowed. Here, we simply compared various models according to the volume of the basin of attraction associated with each equilibrium. An alternative viewpoint would consist of measuring the "strength" with which an equilibrium is attracting the economy. This could be achieved, e.g., by relying on the Freĭdlin–Wentzell theory (see Freĭdlin and Wentzell (1998)). Converting the model into stochastic differential equations by adding a Brownian motion would allow for the study of the probability that a given sample path will remain far from an equilibrium. In other words, we would be able to compute the probability that the system shifts from the basin of attraction of the *good* equilibrium to the *bad* equilibrium. An expected result would be that the probability of straying out of the *good* equilibrium is lower when substitution is higher. The stochastic model would also–through Malliavin calculus–allow for a sensibility test and would help to develop a sturdy understanding of the sensitivity of the model to a change of parameter. This research may strengthen the conclusion of the paper concerning the capability of CES technology to cope with a change of parameter.

Two final extentions are worth mentioning. First, one minght study the structural stability of the dynamical system of each model (CES versus Leontief) in order to test whether the qualitative behavior of the trajectories is affected by C^1-small perturbations. Second, one might analyze possible Hopf bifurcation after introducing lags in the Phillips curve (or in the capital accumulation equation). This change transforms the system from ordinary differential equations to delay differential equations.

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Chapter 5

Coping with the Collapse: A Stock-Flow Consistent Monetary Macrodynamics of Global Warming

Abstract: This paper presents a macroeconomic model of endogenous growth that enables to take into consideration both the economic impact of climate change and the pivotal role of private debt. Using a Goodwin-Keen approach Keen (1995), based on the Lotka-Volterra logic, we couple its nonlinear dynamics of underemployment and income distribution with abatement costs. Moreover, various damage functions *à la* Nordhaus (Nordhaus (2007)) and Dietz-Stern (Dietz and Stern (2015)) reflect the loss in final production due to the temperature increase caused by the rising levels of CO_2 emissions. An empirical estimation of the model at the world-scale enables us to simulate plausible trajectories for the planetary business-as-usual scenario. Our main finding is that, even though the short-run impact of climate change on economic fundamentals may seem *prima facie* rather minor, its long-run dynamic consequences may lead to an extreme downside. Under plausible circumstances, global warming forces the private sector to leverage in order to compensate for output losses; the private debt overhang may eventually induce a global financial collapse, even before climate change could cause serious damage to the production sector. Under more severe conditions, the interplay between global warming and debt may lead to a secular stagnation followed by a collapse in the second half of this century. We analyze the extent to which slower demographic growth or higher carbon pricing allow a global breakdown to be avoided. The paper concludes by examining the conditions under which the $+1.5°C$ target, adopted by the Paris Agreement (2015), could be reached.

JEL Codes:: C51, D72, E12, O13, Q51, Q54

Keywords: Climate change, endogenous growth, damage function, integrated assessment, collapse, stock-flow consistency, Goodwin, debt, secular stagnation.

5.1 Introduction

Taking advantage of over forty years of hindsight available since *The Limits to Growth* was published (Meadows et al. (1972) and Meadows et al. (1974), LtG in the sequel), several attempts to review how society is tracking relative to their ground-breaking modeling have addressed the question of whether the global economy is on a path of sustainability or collapse. Turner (2008), Turner (2012) and Turner (2014) and Hall and Day (2009) (see also Jackson and Webster (2016)) tend to confirm the LtG standard-run scenarios, which forecast a collapse in living standards due to resource constraints in the twenty-first century. The mechanism underlying the simulated breakdown indeed seems consistent with the increasing capital costs of peak oil and net energy (i.e., the decline of energy returned on energy invested, EROI). On the other hand, over a similar time frame, international efforts based around a series of United Nations (UN) conferences have yielded rather mixed results (Linner and Selin (2013) and Meadowcroft (2013)). In these simulations at least, the unraveling of the global economy and environment is essentially due to the growing scarcity of natural resources (energy, minerals, water...), while climate change plays little role, if any.[1] Given the ongoing awareness of climate change damages, crystallized at the diplomatic level in the Paris Agreement of December 2015, this raises the question of whether global warming might *per se* induce a similar breakdown of the world economy.

This paper examines this issue, assuming that the world population will follow the UN median demographic scenario (see World Population Prospects 2015 – Data Bookletuns (2015)). At variance with the literature just mentioned, however, we explicitly model the financial side of the world economy in order to assess the possible negative feedback of private debt on the ability of the world economy to cope with planetary damages. Besides these two basic modifications – our focus on climate warming rather than resource depletion, and on explicit intertwined finance–environment dependencies – the basic

[1]In LtG, of course, scenario 2 simulates a breakdown due to pollution, but the latter does not incorporate the impact of global warming.

mechanism at work turns out to add a new dimension, absent from the narrative initially emphasized by LtG, namely the pivotal role of private debt.

Our own storyline goes as follows: losses due to environmental damages reduce current profits, which are no longer sufficient to fund investment. Yet, investment keeps growing, courtesy of the lending facilities provided by the world banking sector. The lingering level of debt, however, may endanger the world's economic engine itself as it is based on the promise of future wealth creation: whenever the aggravation of climate damages prevents this promise from holding water, the productive sector may indeed become incapable of paying back its debt. Depending on the speed at which labor productivity increases compared to the severity of global warming, the shrinking of investment induced by the burden of private debt may prevent the world economy from further adapting to climate turmoil, leading ultimately to a collapse around the end of the twenty-first century.

In the same way as LtG deliberately leaves the timeline somewhat vague, the main interest of our findings lies, in our view, in the general pattern of behavior they highlight, rather than when exactly particular events might happen.

Finally, the global unraveling captured in this paper can be interpreted as the result of a debt-deflation depression in the sense given to this concept by Fisher (1933). The fact that part of the world economy might be on the verge of falling into a liquidity trap is illustrated today by the two "lost decades" of Japan, of course, but also by the recessionary state of the southern part of the Eurozone, by obstinately negative long-term interest rates on international financial markets and, last but not least, by the brutal contraction of world nominal GDP in 2015 (-6%, Fund (2016)). These paradoxes may be viewed as signals that a secular decline induced by biophysical constraints is seeping the financial sphere. To the best of our knowledge, this paper offers the first macroeconomic narrative where debt-deflation becomes the hallmark of a possible forthcoming breakdown provoked by global warming. It confirms the view defended by Rezai et al. (2013) that policy relevant recommendations should be based on a holistic and macro-perspective. Our hope is that it paves the road toward the kind of Keynesian ecological macroeconomics that was wished for by these authors.

5.1.1 The Dynamics of Debt

Since the financial crisis of 2007-2009, the ideas of Hyman Minsky around the intrinsic instability of a monetary market economy have experienced a significant revival. In the sequel, we adopt a mathematical formalization of Minsky's standpoint in order to assess the role of debt dynamics in our narrative.[2] More precisely, our starting point is the basic Lotka-Volterra dynamics first introduced by Goodwin (1967) and later extended by Keen (1995). Keen (1995) is a three-dimensional non-linear dynamical system describing the time evolution of the wage share, employment rate, and private debt in a closed economy. Under reasonable assumptions, this system admits, among others, two locally stable long-run equilibria: one (the "good" equilibrium) with a finite level of debt and nonzero wages and employment rate, and another (the "bad equilibrium") characterized by and infinite debt-to-output ratio, vanishing wages and zero employment (Grasselli and Lima (2012)). We show how, absent any climatic complications, the world economy would converge towards a "good" steady state. The addition of a climate backloop, modeled through appropriately selected damage and abatement functions, may however drive the state of the economy towards the "bad" long-run equilibrium with unlimited debt, leading to a planetary downside.

Over the past thirty years many integrated assessment models (IAMs in the sequel) have been developed in order to estimate the impact of economic development on the environment. A solid body of literature compares IAM models describing their advantages and disadvantages (Schwanitz (2013)). The models considered in this literature fall into one of four categories, based on the macroeconomic settings they rely on: (1) welfare maximization; (2) general equilibrium; (3) partial equilibrium; and (4) cost minimization (Stanton et al. (2009)). By contrast, our modeling approach is based on a minimal (bounded) rationality requirement on the behavior of imperfectly competitive firms, allows for multiple long-run equilibria, is stock-flow consistent (Godley and Lavoie (2012)), and exhibits endogenous monetary cycles and growth, viscous prices, private debt, and underemployment. Moreover, money is endogenously created by the banking sector (Giraud and Grasselli (2016)) and turns out to be non-neutral (Giraud and Kockerols (2015)). The non-trivial properties of money enables the emergence of a phenomenon such as debt-deflation (Grasselli and Nguyen-Huu (2015)). Here, at variance with general equilibrium approaches (see, e.g., Giraud and Pottier (2016)), debt-deflation need not just appear as a "black swan" – or, more precisely, a "rare" event relegated to the tail of risk distribution. On the contrary, depending on the basin of attraction where the state of the economy is driven by climate damages, the ultimate breakdown may occur as the inescapable consequence of the business-as-usual (BAU) trajectory.

Approaches based on exogenous technology lead to three different types of answers to (some of) these questions depending on their assumptions. Somewhat oversimplifying existing approaches and assigning colorful labels, we can summarize these as follows.

[2] Santos (2005) provides a survey up to 2005 of the literature on the modeling of Minskian instability; more recent contributions include Ryoo (2010) and Chiarella and Di Guilmi (2011).

Nordhaus' answer is that limited and gradual interventions are necessary (see Nordhaus (2007)). Optimal regulation should reduce long-run growth by only a modest amount. Stern's answer (see Stern (2006)) is less optimistic. It calls for more extensive and immediate interventions, and argues that these interventions need to be in place permanently even though they may entail significant economic cost. The more pessimistic Greenpeace answer is that essentially all growth needs to come to an end in order to save the planet.

The answer provided by the thought experiments run in this paper clearly stands on the side of Stern's viewpoint: as we shall show in companion papers, fundamental bifurcations led by strong interventions may prevent the economic world from a disaster but they are not detailed here. Rather, we focus on the business-as-usual perspective, and show that gradual and marginal interventions will not suffice: too narrow an answer to the climate challenge may lead to an end of growth by disaster (not by design) and to forced degrowth.

5.1.2 The Climate and Economy Interaction

By contrast with the literature based on the Ramsey-Cass-Koopmans model, we incorporate endogenous drivers of growth and allow climate change to damage these drivers. As argued indeed by Stern (Stern (2013)), climate change could have long-lasting impacts on growth. We borrow from the emerging body of empirical evidence pointing in this direction (e.g., Dell et al. (2012)), even though climatic conditions in the recent past have been relatively stable compared with what we now have to contemplate in the near future – which suggests that the "real story" might be even worse than what we are able to forecast.

Second, we consider various types of convexity of the damage function linking the increase in global mean temperature with the instantaneous reduction in output. That it might be highly convex at some temperature is strongly suggested by the literature on tipping points (see Dietz and Stern (2015) and Weitzman (2012)). By contrast, most existing IAM studies assume very modest curvature of the damage function. The DICE default, for instance, is quadratic, and our simulations confirm that it leads to unrealistic narratives (see Section 5.4.1 below).

Third, following Dietz and Stern (2015) and Weitzman (2012), we allow for explicit and large climate risks by considering the possibility of high values of the climate-sensitivity parameter (i.e., the increase in global mean temperature, in equilibrium, accompanying a doubling in the atmospheric concentration of CO_2). We conduct sensitivity analysis on high values, but also rely on a probability distribution reflecting the latest scientific knowledge on the climate sensitivity as set out in the recent IPCC report (Stocker et al. (2013)). Its key characteristic is a fat tail of very high temperature outcomes that are assigned low probabilities. By contrast, most IAM studies have ignored this key aspect of climate risk by proceeding with a single, best-guess value for the climate sensitivity, typically corresponding to the mode of the IPCC distribution. Even though the Intentional Nationally Determined Contributions reported by the Parties to the Paris Agreement (see also Stocker et al. (2013)) should induce an average increase in temperature of 3.5°C by the end of this century, it is known that they are compatible with a 10% probability of reaching +6°C. We show that the path leading to such a level of warming would lead to a planetary collapse in the second half of this century. As written by Dietz and Stern (2015): "this is not just a 'tail' issue."

The brief overview of collapses provided, e.g., in (Motesharrei et al. (2014), p.91) not only shows the ubiquity of cycles of rise-and-collapse, but also the extent to which advanced, complex, and powerful societies are susceptible to collapse, even precipitously: "The fall of the Roman Empire, and the equally (if not more) advanced Han, Mauryan, and Gupta Empires, as well as so many advanced Mesopotamian Empires, are all testimony to the fact that advanced, sophisticated, complex, and creative civilizations can be both fragile and impermanent." In the thought experiment suggested in Motesharrei et al. (2014), the authors argue that the Lotka-Volterra dynamics might be the secret dynamical invariant explaining this seemingly universal process of overshoot and collapse. Here, we partially confirm this suggestion by showing that the prey-predatory dynamics, when properly introduced into a macroeconomic framework through the interplay between debt, employment, and wages, leads to a similar conclusion: the world economy, as we know it today, is not immune to such a fate.

The paper is organized as follows: Section 2 sets the scene by introducing the stock-flow consistent macroeconomic model in presence of damage and abatement costs. In Section 3, the climate module is presented, describing the interconnection between the output level, emissions, CO_2 concentration, average atmosphere temperature increase, and damages induced by climate change. Section 4 discusses the different scenarios arising from the interplay of our various key parameters. The final section summarizes the conclusions and outlines areas for future research.

5.2 Monetary macrodynamics

Our underlying macroeconomic model closely follows the contribution of Grasselli and Lima (2012) and the literature centered around Keen (1995)'s approach such as Grasselli and Lima (2012), Grasselli et al. (2014) andGrasselli and Nguyen-Huu (2015) and Nguyen-Huu and Costa-Lima (2014) among others. This framework, based on a Lotka-Volterra logic, is motivated by the aftermath of the 2008 sub-prime mortgage crisis, during which private debt played a pivotal role in endangering the world's macroeconomic stability. One appeal of this literature lies in its ability to formalize economic collapse as a consequence of overindebtedness. We depart from this literature, however, by endogenizing labor productivity growth, and add to the resulting structure climate-change feedback including temperature, abatement costs, and a damage function.

5.2.1 The Basics

Absent any damages due to climate change, the "gross" real output, Y^*, is linked to the stock of productive capital, K, by a linear relationship, where, for simplicity, the capital-output ratio, $\nu > 0$, is assumed to be constant,

$$Y^* := \frac{K}{\nu}.$$

In this paper, $\nu \simeq 2.89$.[3] Introducing a damage function as in Nordhaus and Sztorc (2013), current production is cut so that the global supply diminishes by the quantity $\mathbf{D}\frac{K}{\nu}$, induced by global warming. Real output, Y, becomes

$$Y := (1 - \mathbf{D})\frac{K}{\nu}. \tag{5.1}$$

The damage function, $\mathbf{D}$, is an increasing nonlinear function of the global atmospheric temperature deviance relative to its value, T, in 1900, defined shortly.

Let D denote the nominal private debt of firms. Its evolution depends upon the gap between nominal investment, pI, *plus* nominal dividends paid to the firms' shareholders, D_i, and current nominal profit, Π, that is,

$$\dot{D} := pI + D_i - \Pi, \tag{5.2}$$

[3] See Appendix D.5.1 for details. The extension to an endogenous capital-output ratio is left for further research.

where p is the current price of consumption. The current nominal profit, Π, is

$$\Pi := pY - W - rD,$$

with W being the nominal wage bill, and r the (constant) short-term nominal interest rate.[4] Denoting the total workforce by N, and the number of employed workers by L, the productivity of labor, the employment rate, and the nominal wage per capita are given respectively by

$$a := \frac{Y^*}{L}, \quad \lambda := \frac{L}{N} \quad \text{and,} \quad w = \frac{W}{L}. \tag{5.3}$$

Denoting $\omega := wL/pY$ the wage share, and $d := D/pY$ the private debt ratio, net profit becomes $\Pi = (1 - \omega - rd)pY$. In the sequel, we denote by

$$\pi := 1 - \omega - rd \tag{5.4}$$

the net profit share. Capital accumulation obeys the standard equation, expressed in real terms as

$$\dot{K} := I - \delta K,$$

where δ is the constant depreciation rate of capital. Net investments are equal to gross investments minus abatement costs (defined shortly) paid by investors as in Nordhaus and Sztorc (2013), that is,

$$I := Y(\kappa(\pi) - \mu G), \tag{5.5}$$

where $\kappa(\cdot)$ is an increasing function of π, $G \in [0, 1]$ denotes the real abatement costs imposed on the economy, and $\mu \in [0, 1]$ is measuring the fraction of abatement costs paid by investors. The function $\kappa(\cdot)$ will be empirically estimated.[5]

Here, as in both the Goodwin and Keen models, the behavior of households is fully accommodating in the sense that, given investment, consumption is determined by the macro balance

$$C := Y - I = (1 - \kappa(\pi))Y,$$

precluding more general specification of households saving propensity.[6]

[4]Here, for simplicity, r is kept constant. We shall endogenize it and analyze in depth the impact of Taylor rules in a subsequent paper.

[5]See Appendix D.5.4 for details. We refrain from providing micro-foundations to either $\kappa(\cdot)$ or $\Phi(\cdot)$. Indeed, as shown by Mas-Colell et al. (1995), full-blown rational corporates, when they are sufficiently numerous, are exposed to an "everything-is-possible" theorem *à la* Sonnenschein-Mantel-Debreu at the aggregate level. Our phenomenological approach allows for this kind of emergence phenomena.

[6]Studying the consequences of dropping Say's law will be the task of a forthcoming paper.

Observe that equation (5.5) follows the conventional linkage of damages with output, as first formalized by Nordhaus (1993) and Stern (2006). It precludes any "rebound effect" in the sense recently highlighted by Taylor and other ecological economists Rezai et al. (2013) and Taylor et al. (2016). As stressed by these authors, indeed, an increase in expenditure on mitigation may boost output, and thereby GHG emissions, in a macroeconomic analog of the "Jevons paradox" or the "rebound effect". By contrast, equation (5.5) implies that mitigation *substitutes* to otherwise standard investment. As we shall see in the sequel, this forced substitution creates a number of problems —to begin with the lack of proper investment, thereby the lack of growth capable of funding both standard expenditures and abatement costs. The dual viewpoint (where abatement costs are aditional to normal investment) will be dealth with in a companion paper.

Table 5.1 makes explicit the stock-flow consistency of our model.

	Households	Firms		Banks	Sum
Balance Sheet					
Capital stock		pK			pK
Deposits	M^h	M^f		$-M$	
Loans		$-L$		L	
Equities	E^h	$-E^h$			
Sum (net worth)	X^h	X^f		X^b	X
Transactions		current	capital		
Consumption	$-pC$	pC			
Investment		pI	$-pI$		
Accounting memo [GDP]		$[pY]$			
Wages	W	$-W$			
Profits	Di	$-\Pi$	$\Pi - Di$		
Interests on deposits	rM^h	rM^f		$-rM$	
Interests on loans		$-rL$		rL	
Financial Balances	S^h	0	$-\dot{D}$	S^b	
Flow of funds					
Gross Fixed Capital Formation		pI			pI
Change in Deposits	$\dot{M}^h$	$\dot{M}^f$		$-\dot{M}$	
Change in loans		$-\dot{L}$		$\dot{L}$	
Change in equities	$\dot{E}^h$	$-\dot{E}^h$			
Column sum	S^h	$\Pi - Di$		S^b	pI
Change in net worth	$\dot{X}^h = S^h$	$\dot{X}^f = \Pi - Di + (\dot{p} - (\delta + \frac{\mu}{\nu}G)pK$		$\dot{X}^b = \Pi^b$	$\dot{X}$

TABLE 5.1: Balance sheet, transactions, and flow of funds in the economy

It can be readily checked that, in this set-up, the accounting identity "investment = saving" always holds. This model departs from Grasselli and Nguyen-Huu (2015) in four ways. First, it includes damages induced by climate change. Second, we make explicit the dividends paid by firms to households. Third, as we shall see *infra*, labor productivity growth will be endogenous. Fourth, the labor force is no longer assumed to grow exponentially but rather according to a sigmoid inferred from the 15–64 age group in the United Nations scenario, World Population Prospects 2015 – Data Booklet.uns (2015)

$$\frac{\dot{N}}{N} := q\big(1 - \frac{N}{M}\big),$$

where $M \approx 7.056$ billion is the upper bound of the labor force on Earth and q is the speed of convergence towards M (i.e., basically, the pace at which the demographic transition is assumed to take place in sub-Saharan Africa).[7]

5.2.2 Endogenous Monetary Business Cycles

The link between the real and nominal spheres of the economy is provided by a short-run wage-price dynamics taken from Grasselli and Nguyen Huu (2016).[8]

$$\frac{\dot{w}}{w} := \phi(\lambda) + \gamma i, \tag{5.6}$$

where $0 \leq \gamma \leq 1$, $\eta_p > 0$, $m \geq 1$, and

$$i = \frac{\dot{p}}{p} := -\eta_p\Big(1 - m\frac{w}{(1-\mathbf{D})pa}\Big) + c = \eta_p(m\omega - 1) + c. \tag{5.7}$$

Equation (5.6) states that workers bargain for wages based on the current state of (un)employment (as in Keen (1995)), but also take into account the observed inflation rate, i. The constant γ measures the degree of monetary illusion, with $\gamma = 1$ corresponding to the case where workers fully incorporate inflation into their bargaining (no "money illusion"). Equation (5.7) captures the dynamics of inflation, where the long-run equilibrium price is given by a markup, $m \geq 1$, times the unit labor cost, $W/pY = w/pa$. Observed prices converge to this long-run target through a lagged adjustment of exponential form with a relaxation time $1/(\eta_p - c)$.[9] Whenever the consumption goods market is imperfectly competitive, $m > 1$.

The real growth rate, g, of the economy can be expressed as a function of the growth rate of capital:

$$g := \frac{\dot{Y}}{Y} = \frac{\dot{K}}{K} - \frac{\dot{\mathbf{D}}}{1-\mathbf{D}}. \tag{5.8}$$

[7]The estimation of the vector (q, M) is detailed in Appendix D.3.

[8]See also Mankiw (2010) for a justification of the short-run Philips curve.

[9]The estimation of these parameters is available in Appendix D.5.7. It plays a role analogous to the Calvo parameter in the neo-Keynesian literature, capturing the viscosity of prices.

On the other hand, since $Y = (1 - \mathbf{D})aL$, it follows that

$$g = \frac{\dot{a}}{a} + \frac{\dot{L}}{L} - \frac{\dot{\mathbf{D}}}{(1 - \mathbf{D})} \tag{5.9}$$

Equations (5.8) and (5.9) illustrate the impact of climate change on real growth. By means of illustration, whenever labor productivity grows exogenously at the rate $\dot{a}/a = \alpha > 0$, the wage share, $\omega = w/p(1 - \mathbf{D})a$ evolves according to

$$\frac{\dot{\omega}}{\omega} = \frac{\dot{w}}{w} - \frac{\dot{a}}{a} + \frac{\dot{\mathbf{D}}}{1 - \mathbf{D}} - i \tag{5.10}$$

$$= \phi(\lambda) - \alpha + \frac{\dot{\mathbf{D}}}{1 - \mathbf{D}} - (1 - \gamma)i. \tag{5.11}$$

Changes in the employment rate are given by

$$\begin{aligned} \frac{\dot{\lambda}}{\lambda} &= \frac{\dot{L}}{L} - \frac{\dot{N}}{N} = \frac{\dot{Y}}{Y} - \frac{\dot{a}}{a} - \frac{\dot{N}}{N} + \frac{\dot{\mathbf{D}}}{(1 - \mathbf{D})} \\ &= g - \alpha - q\left(1 - \frac{N}{M}\right) + \frac{\dot{\mathbf{D}}}{(1 - \mathbf{D})}. \end{aligned}$$

It is worth mentioning that, in the short run, a more severe damage process will favor some redistribution of income toward workers (see equation 5.10). As we shall see, however, this impact is limited since, above a certain threshold, global warming induces an unraveling that may lead in the long-run to full unemployment and a zero wage share.

Absent climate damages (i.e., whenever $\mathbf{D} = 0$), equations (5.1) and (5.3) can be viewed as arising from a Leontief production function

$$Y^* = \min\left\{\frac{K}{\nu}, aL\right\},$$

together with a minimal rationality requirement: $\frac{K}{\nu} = aL$ along any trajectory, which says that the productive sector works at full capacity and does not hire more employees than needed, given the level of gross real output permitted by installed capital.

Finally, the prey-predatory forces underlying our dynamics are best viewed in the simple case of exponential technological progress without climate change feedback, summarized by a four-dimensional, non-linear dynamical system, where div $:= D_i/pY$ is the constant share of dividend per nominal GDP distributed to households by the non-financial private sector:[10]

$$
\begin{cases}
\dot{\omega} & = \omega\left[\phi(\lambda) - \alpha - (1-\gamma)i\right] \\
\dot{\lambda} & = \lambda\left[\frac{\kappa(\pi)}{\nu} - \delta - \alpha - \frac{\dot{N}}{N}\right] \\
\dot{d} & = d\left[r - \frac{\kappa(\pi)}{\nu} + \delta - i\right] + \kappa(\pi) - (1-\omega) + \text{div} \\
\dot{N} & = qN\left(1 - \frac{N}{M}\right),
\end{cases}
$$

which is easily seen to embed a Kolmogorov type of predator-prey model (i.e., a generalized autonomous Lotka-Volterra system, see Brauer and Castillo-Chavez (2000)), where ω is the predator, λ is the prey, $\frac{\partial\phi}{\partial\lambda} > 0$, and, given equation (5.4), $\frac{\partial\kappa}{\partial\omega} < 0$ as soon as $\kappa(\cdot)$ is increasing. Figure 5.1 provides a typical diagram phase in the (ω, λ, d) space.

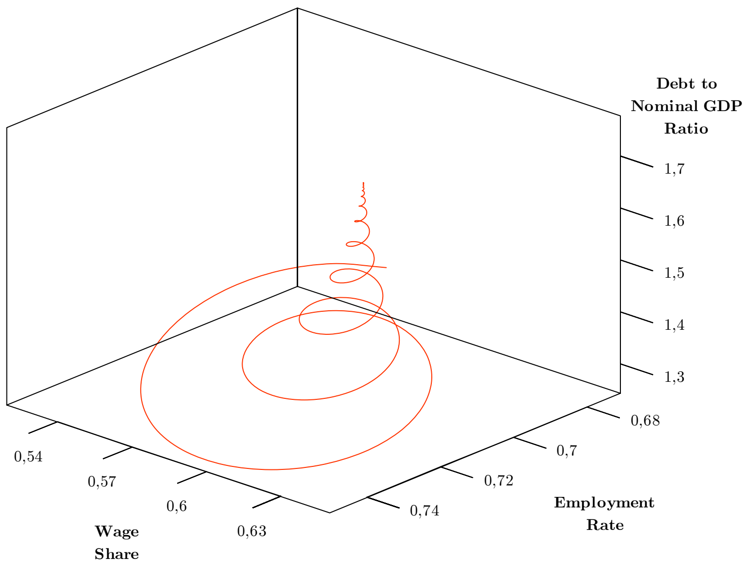

FIGURE 5.1: Phase diagram of employment rate *vs.* wage share and debt ratio in the exponential case.

As output grows, more workers are employed, hence the employment rate increases, which eases labor negotiations and, courtesy of the short-run Phillips curve (equation 5.6), induces an increase of the wage share, ω. As a result, inflation tends to accelerate (equation 5.6), which induces a positive backloop on wages as soon as workers do not share complete monetary illusion (i.e., $\gamma > 0$). As shown by equation 5.4, however, this process will devour the profit share, π, hence reducing investment (see equation 5.5).

[10]See Appendix D.5 for its calibration.

The slowdown of capital growth then results in a lower growth rate of output, reducing the growth rate of employment (remember equation 5.9). This reversal in trend cools down the wage growth rate, restoring the profit rate in the medium run and hindering cost-push inflation. Next, if the higher profitability induces a sufficiently strong increase in investment, aggregate demand can overcome initial, relative reduction in consumption due to the redistribution of income from workers to investors, so that output growth goes up again. According to Taylor et al. (2016), this "paradox of thrift" characterizes profit-let adjustments in high-income countries. The issue at stake in this paper is to investigate how far climate damages may perturb such a virtuous cyclical behavior by preventing profitability from boosting investment.

Empirically estimated at the world level, this simplistic version of our model yields an endogenous cycle with a periodicity of 12-18 years – thus close to the Kuznets business swings (cf. Kuznets (1930)). In the long run, however, the magnitude of each cycle shrinks, and the state of the economy converges towards a stationary state: while output still grows exponentially, the endogenous volatility of most parameters tends to zero, and a phenomenon akin to the "Great Moderation" occurs. The employment rate oscillates around 72%, and the wage share converges in the region of 0.62. At variance, however, with the infamous Great Moderation observed in the decade preceding the global financial crises of 2007-2009, here, the economy converges to a *bona fide* long-run steady state since the debt-to-output ratio also stabilizes at around 1.71. Moreover, both this equilibrium and the "bad steady state" with infinite debt ratio turn out to be locally stable, given our empirical estimation and the analytical conditions found in Grasselli and Lima (2012).

The main purpose of this paper is to understand how global warming will affect this basic cyclical interplay between real and monetary forces.

5.3 The Climate Module

This section presents the climate feedback on the economy, that is to say, CO_2 emissions, their impact on temperature, and the damage their build-up causes to real output. This module is inspired by its analog in Nordhaus and Sztorc (2013).

5.3.1 CO_2 Emissions

As in Nordhaus (2014), global CO_2 emissions are the sum of two terms: (i) E_{ind}, the industrial emissions linked to the consumption of fossil energies and (ii) E_{land}, the land-use emissions:[11]

$$E := E_{ind} + E_{land}.$$

Industrial emissions are endogenously determined and depend on the level of real output according to

$$E_{ind} := Y\sigma(1-n),$$

where n is the emission-reduction rate consequent to abatement efforts, and σ is the emission intensity of the economy. The latter is assumed to behave according to:

$$\frac{\dot{\sigma}}{\sigma} := g_\sigma, \text{ with } \frac{\dot{g_\sigma}}{g_\sigma} := \delta_{g_\sigma},$$

where $\delta_{g_\sigma} < 0$. While the initial value of σ is empirically given by data, the initial value of g_σ and the calibration of δ_g are set as in Nordhaus and Sztorc (2013) to ensure that $g_\sigma \simeq -0.95\%$ per year until 2100 and -0.87% up to 2200. The dynamics of n will be presented shortly.

Land-use emissions are exogenously given:

$$\frac{\dot{E}_{land}}{E_{land}} := \delta_E,$$

with $\delta_E < 0$. As Nordhaus and Sztorc (2013), the calibration is based on results from the Fifth Assessment of the Stocker et al. (2013), which reports a 3 $GtCO_2$ per annum contribution of land-use changes.

5.3.2 CO_2 Accumulation

The carbon cycle is modeled through the interaction between three layers in which total CO_2 emissions, E, accumulate: (i) the atmosphere (AT); (ii) a mixing reservoir in the upper ocean and the biosphere (UP) and; (iii) the deep ocean (LO). This mechanism is represented by the matrix system

$$\begin{pmatrix} \dot{CO_2}^{AT} \\ \dot{CO_2}^{UP} \\ \dot{CO_2}^{LO} \end{pmatrix} = \begin{pmatrix} E \\ 0 \\ 0 \end{pmatrix} + \underbrace{\begin{pmatrix} -\phi 12 & \phi 12 \frac{C_{AT_{eq}}}{C_{UP_{eq}}} & 0 \\ \phi 12 & -\phi 12 \frac{C_{AT_{eq}}}{C_{UP_{eq}}} - \phi 23 & \phi 23 \frac{C_{UP_{eq}}}{C_{LO_{eq}}} \\ 0 & \phi 23 & -\phi 23 \frac{C_{UP_{eq}}}{C_{LO_{eq}}} \end{pmatrix}}_{:=\Phi} \begin{pmatrix} CO_2^{AT} \\ CO_2^{UP} \\ CO_2^{LO} \end{pmatrix},$$

[11] In concrete terms, this second term can be viewed as being induced by deforestation and the implied release of CO_2.

where E stands for the total CO_2 emissions, CO_2^i is the CO_2 concentration in layer $i \in \{AT, UP, LO\}$, ϕ_{ij} captures the CO_2 flow from layer i to layer j, and $C_{i_{eq}}$ is some constant scaling parameter corresponding to the pre-industrial CO_2 equilibrium concentration on layer i.

At first, total emissions E are released into the atmospheric layer increasing its CO_2 concentration. Then, through diffusion and absorption phenomena, they spread between the other layers according to the matrix Φ. Note that each column of Φ sums to zero, meaning that the total CO_2 concentration in all three layers is accumulating at the rate of emissions E. As a result, assuming the atmospheric layer is the sole contributor to radiative forcing, the remaining layers act as sinks and mitigate temperature increase over time.[12]

5.3.3 Radiative Forcing

The accumulation of greenhouse gases from anthropogenic sources induces a change, F, in global radiative forcing according to

$$F := F_{ind} + F_{exo},$$

where F_{ind} stands for the radiative forcing due to CO_2 accumulation (closely linked to industrial production, and projected by the Stocker et al. (2013) to be the main contributor of global warming), and F_{exo} stands for an exogenous forcing capturing the impact of other long-lived greenhouse gases and other factors such as albedo changes or the cloud effect. According to the Representative Concentration Pathways (hereafter RCP) database provided by the Fifth Assessment of the Stocker et al. (2013), the effect of non-CO_2 radiative forcing is estimated to be lower than CO_2 radiative forcing. Therefore, following Nordhaus and Sztorc (2013), exogenous forcing will be modeled by:

$$\dot{F}_{exo} = \delta_{F_{exo}} F_{exo} \left(1 - \frac{F_{exo}}{0.7}\right)$$

with $\delta_{F_{exo}} > 0$. In 2010, the exogenous forcing is calibrated at 0.25 W/m^2 and designed to grow smoothly in order to be close to 0.7 W/m^2 in 2100, in line with the RCP trajectories.

[12]The calibration of the matrix Φ is available in Appendix D.4.

Industrial radiative forcing is defined as follows:

$$F_{ind}(t) = \frac{F_{2\times CO_2}}{\log(2)} \log\left(\frac{C_{CO_{2(t)}}}{C_{CO_{2(t0)}}}\right),$$

where F_{2CO2} is the net radiative forcing associated with a doubling of atmospheric CO_2 concentration.

5.3.4 Temperature Change

A change of radiative forcing induces a change, T, in the global mean atmospheric temperature. Following Geoffroy et al. (2013), global thermal behavior is modeled through an energy-balance model of two coupled layers: (i) the atmosphere, land surface, and upper ocean, and (ii) the deeper ocean. In this framework, each layer obeys:

$$\begin{cases} C\dot{T} = F - (RF)T - \gamma^*(T - T_0) \\ C_0\dot{T}_0 = \gamma^*(T - T_0), \end{cases} \tag{5.12}$$

where F is the radiative forcing, RF is the radiative feedback parameter, γ^* is the heat exchange coefficient, C is the heat capacity of the atmosphere, land surface and upper ocean layer, C_0 is the heat capacity of the deep ocean layer and T and T_0 are the mean temperature perturbation (deviation from the 1900 value) of respectively the atmosphere, land surface and upper ocean layer, and the deep ocean layer. The long-run equilibrium of the temperature anomaly is given by $T = F/RF$ and will control the climate sensitivity as in Dietz and Stern (2015).[13]

5.3.5 Climatic Damages

For our baseline scenario, we rely on the quadratic form of the damage function proposed by Nordhaus (2014):

$$\mathbf{D} = 1 - \frac{1}{1 + \pi_1 T + \pi_2 T^2},$$

with $\pi_1 = 0$, $\pi_2 = 2.84 \times 10^{-3}$. Initially, the function relied on various sectoral studies such as crop losses or change in energy demand for space cooling or heating for several points of global warming. It was then aggregated to describe the global impact of global warming based on the estimates of Tol (2009). This damage function is explicitly designed to model the effects of a global warming contained within a range of 0°C to

[13]The calibration of the system (5.12) is available in Appendix D.2.

3°C. No threshold effects are thus allowed in this scenario after a temperature anomaly above 3°C.

5.3.6 Abatement Costs

As public policy instruments are deployed, carbon emission abatement is achieved with a cost, GY, partly borne by investment at the rate μ.[14] The ratio, G, of abatement cost to real output is defined by the following relation:

$$G = \theta_1 n^{\theta_2},$$

where n stands for the rate of emissions reduction, defined shortly, and $\theta_1, \theta_2 > 0$.[15] Here, $\theta_2 > 0$ is calibrated so that the abatement-cost-to-output ratio is highly convex.

The dynamics of abatement costs is linked to a carbon price instrument p_C[16]and to a backstop technology with a price p_{BS} —able to replace carbon-intensive technology. These prices follow exogenous trajectories defined by

$$\frac{\dot{p}_{BS}}{p_{BS}} = \delta_{p_{BS}} < 0,$$

$$\frac{\dot{p}_C}{p_C} = \delta_{p_C} > 0.^{17}$$

The reduction rate reflects an arbitrage relation between the relative prices of carbon and of the backstop technologies respectively,

$$n := \min\left\{\left(\frac{p_c}{p_{BS}}\right)^{\frac{1}{\theta_2 - 1}}; 1\right\}.$$

The parameter, θ_1, reflects the cost of investing in the backstop technology through its price and the carbon intensity of the economy. The parameters θ_1, θ_2 are calibrated according to Nordhaus and Sztorc (2013). Whenever $p_c \geq p_{BS}$, the energy shift is completed.

[14]For the sake of completeness, the remaining part, $(1-\mu)$, is borne by households. However, in the sequel, we will consider a unitary value for μ.

[15]We follow the calibration of Nordhaus and Sztorc (2013) for (θ_1, θ_2) with $\theta_1 = \frac{\sigma p_{BS}}{1000\theta_2}$ and $\theta_2 = 2.8$.

[16]p_C refers to the price per ton of CO_2.

[17]A sensitivity analysis will be provided. First, we use the same baseline as in Nordhaus and Sztorc (2013). The next scenarios, based on Nordhaus and Sztorc (2013) and Dietz and Stern (2015), will be calibrated so that the growth parameter δ_{p_C} and the intial value of P_C is in line with the optimal values reported in 2020 and 2050 (resp. 2015 and 2055), defined shortly.

Having described all the ingredients of our model, we are now ready to analyze the scenarios resulting from the interaction between global warming and debt accumulation.

5.4 Scenarios

Using the calibration depicted *supra*, we are now ready to discuss four main classes of narratives that depend on labor productivity growth scenarios and previously defined damage functions. The baseline case, the influence of each specification, and finally some combinations are successively analyzed. For each scenario, the initial point is the year 2010, due to data availability constraints as detailed in Appendix D.1, and the simulations run until 2300 —unless a collapse occurs before the final model period. The latter time horizon is in line with Nordhaus and Sztorc (2013) and compatible with the achievement of the energy shift, and thus the beginning of the decline of damages.

5.4.1 Exponential Technological Progress

For the baseline case, we begin with an exogenous technological progress dynamic. Labor productivity is assumed to grow at a constant rate:

$$\frac{\dot{a}}{a} := \alpha > 0. \tag{5.13}$$

For our simulations, we shall adopt two values for α: either 0.0226 or 0.015. The first is based on an observed trend of technological progress at the world level.[18] The second is a proxy of the parameterization adopted in Nordhaus and Sztorc (2013). Table 5.2 presents the main parameters of our baseline simulation, obtained in the "exponential case" $\alpha = 0.0226$, as specified in Appendix (D.5.2). Monetary values are in US$ trillions. The population size is in billions. The GDP deflator (price level) is normalized to 1 in 2010. As already said, the short-run interest rate, r, is kept constant at 3%.

Parameter	Y_{2010}	N_{2010}	ω_{2010}	λ_{2010}	d_{2010}	p_{2010}	α
Value	64.4565	4.5510	0.5849	0.6910	1.4393	1	0.0226

TABLE 5.2: Main macroeconomic parameters of the exponential case

Figure 5.2 presents the deterministic exponential trajectories of our main macroeconomic and climate variables. After some fluctuations, the world economy reaches a stable path with a finite debt ratio and stable inflation (roughly 2%). Yet, due to high emissions of

[18] See details in Appendix D.5.2.

CO_2 (up to approximately 147.34 Gt CO_2 in 2100), the temperature increases (3.95°C in 2100 in the atmospheric layer) and thus augments damages to production. As a consequence, in the (ω, λ, d)-space represented in Figure 5.4, the wage share, ω, and the employment rate, λ, slowly decrease. Then, as the energy shift reaches completion a little before 2250, CO_2 concentration, and thus damages, decrease.

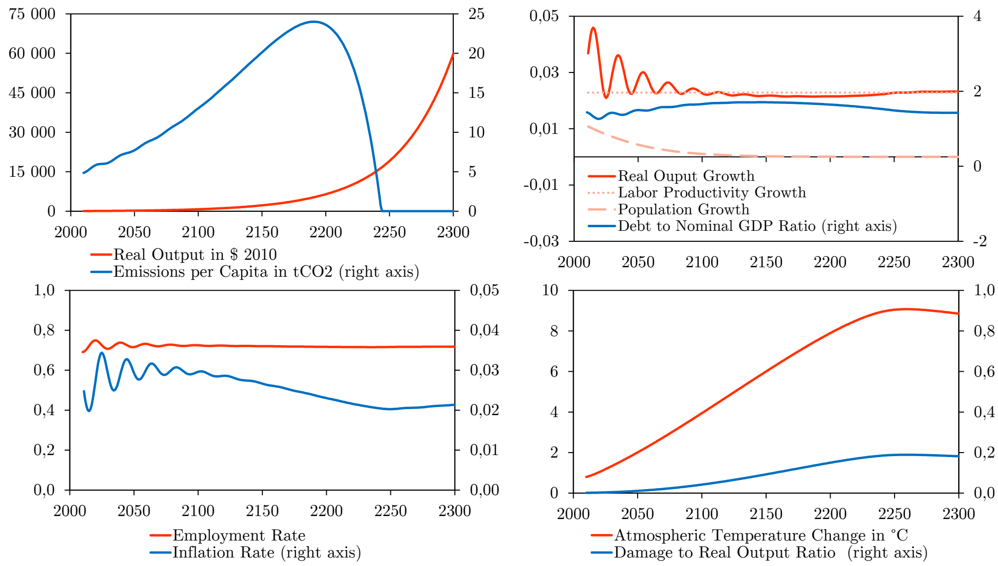

FIGURE 5.2: Trajectories of the main simulation outputs in the exponential case.

This baseline scenario yields important takeaways. First, as could be expected, deterministic exponential productivity growth successfully drives the exponential growth of real GDP, despite climate damages. In 2100, it reaches 11.5 times its initial volume in 2010. This uninterrupted growth is accompanied by monetary and real cycles with a periodicity of 12-17 years. This is consistent with the celebrated Kuznets swings, which have been recently re-examined by Korotayev and Tsirel (2010). However, as time passes, these cycles seem to shorten during the first half of the twenty-second century. Ultimately, all volatility vanishes. Such a secular "Great Moderation" means that the global dynamical system is converging towards a long-run equilibrium where the employment rate stabilizes around a comfortable 72% ratio, while the real output growth rate converges slightly above 2%. The (private nonfinancial corporate) debt-to-GDP ratio reaches a maximum in the middle of the twenty-second century (around 1.7) before declining towards 142%.

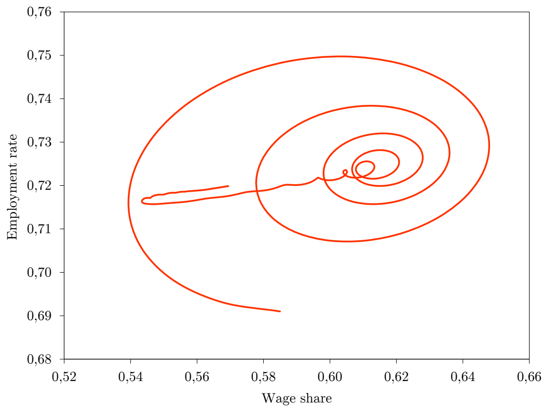

FIGURE 5.3: Phase diagram of employment rate vs. wage share in the exponential case over the period 2010-2900.

As a consequence of the rise in CO_2 concentration, the state of the world economy deviates from its long-run stationary point around 2150 —as illustrated by the phase portrait (ω, λ) in Figure 5.3. This temporary deviation reflects the impact of damages on the long-run equilibrium and especially on the wage share. Indeed, from 2100 until the energy shift is completed, the rise in damage, $\dot{\mathbf{D}}$, is at its highest. At the same time, the world population is plateauing so that demography no longer contributes to output growth. The latter is uniquely driven by α, the growth rate of labor productivity. Output, however, is penalized by climate change, so that its real growth remains below its potential: $g < \alpha$. As a consequence, the employment rate, λ, slightly declines. Indeed, $L = Y/(a(1 - \mathbf{D}))$, so that

$$\frac{\dot{\lambda}}{\lambda} = \frac{\dot{L}}{L} = g + \frac{\dot{\mathbf{D}}}{(1 - \mathbf{D})} - \alpha.$$

Since

$$g = \frac{(\kappa(\pi) - \mu G)(1 - \mathbf{D})}{\nu} - \delta - \frac{\dot{\mathbf{D}}}{(1 - \mathbf{D})},$$

we get

$$\frac{\dot{L}}{L} = \frac{(\kappa(\pi) - \mu G)(1 - \mathbf{D})}{\nu} - \delta - \alpha < 0 \quad \text{or close to } 0.$$

Meanwhile, the wage share, ω decreases as a consequence of three forces conspiring together:

$$\frac{\dot{\omega}}{\omega} = \frac{\dot{w}}{w} + \frac{\dot{\lambda}}{\lambda} - i - g.$$

where i stands for inflation. Now, the decline in λ (with respect to its long-run stationary value) cuts down wages via the short-run Phillips curve, and $\dot{\lambda}/\lambda < 0$, while $i, g > 0$. As soon as damages reduce courtesy of the (very slow) atmospheric cooling (around 2225), employment goes up again, and money wages start growing back. Hence the (slow) return of the world economy to its long-run steady state.

FIGURE 5.4: Phase diagram of employment rate vs. wage share and debt ratio in the exponential case.

The three-dimensional phase diagram in Figure 5.4, shows how the "large deviation" between 2150 and the twenty-fourth century is accompanied by a (temporary) mild deleverage of the production sector. the wage share shifts towards 0.55.

This first scenario offers quite a reassuring picture in the time scale of the century. By 2050, as shown in Table 5.3, the average yearly CO_2 emission per capita is 7.72t. The temperature change in 2100 is +3.94°C and CO_2 concentration, 968.98 ppm. Despite the fact that we are far above the goal unanimously adopted at the Paris Agreement in 2015, the world economy is going pretty well: the damages induced by global warming reduce the final world real GDP by one fifth – a fraction higher than the 5% losses first envisaged by Stern (2006) but this is seemingly easily counterbalanced by the strength of the postulated exogenous growth. As a result of this quite unrealistic picture, CO_2 emissions peak only in the middle of the twenty-second century and the zero-emission level is reached one century later!

GDP Real Growth 2100 (wrt 2010)	1053%
t CO_2 per capita (2050)	7.72
Temperature change in 2100	+3.94 °C
CO_2 concentration 2100	968.98 ppm

TABLE 5.3: Key values of the world economy by 2100 – the exogenous case.

By exhibiting a relatively low impact of damages and negligible abatement costs (less than 1% of real output) for production by 2100 or so, this scenario above all confirms the unrealistic feature of the climate-economy interaction modeling on which it is based. As we shall now see, the picture changes dramatically as soon as labor productivity is made endogenous.[19]

5.4.2 Endogenous Productivity

Let us now discuss alternative scenarios of endogenous labor productivity growth combined with the damage function introduced by Nordhaus and Sztorc (2013).

5.4.2.1 The Kaldor-Verdoorn Case

The Kaldor-Verdoorn case assumes a relationship between labor productivity growth and output growth (cf. Verdoorn (2002)) in the form

$$\frac{\dot{a}}{a} := \alpha + \eta g, \tag{5.14}$$

with g being the real output growth, and $\alpha, \eta > 0$. Equation 5.14 can be interpreted as reflecting dynamic economies of scale (or "learning by doing"). A rough estimate for the United States over the last four decades is $\alpha \simeq 0$ and $\eta \simeq 0.5$,[20] with a tendency of both parameters to increase due to the impact of Information and Communication Technologies. In our simulations, we assume equation 5.14 to hold at the world level. This will be considered optimistic or pessimistic depending on how strongly one believes that the opportunity for emerging economies to follow a learning process analogous to the recent trend in the United States is realistic or not.

[19]For the sake of comparison with DICE, the "Nordhaus scenario", and the "Gordon (2014) scenario" – where labor productivity grows approximately at the deterministic pace of 1.5% and 1.3% respectively – are discussed in Appendix D.6. The findings are qualitatively similar to the ones just described.

[20]See Vernengo and Berglund (2000) for these estimates.

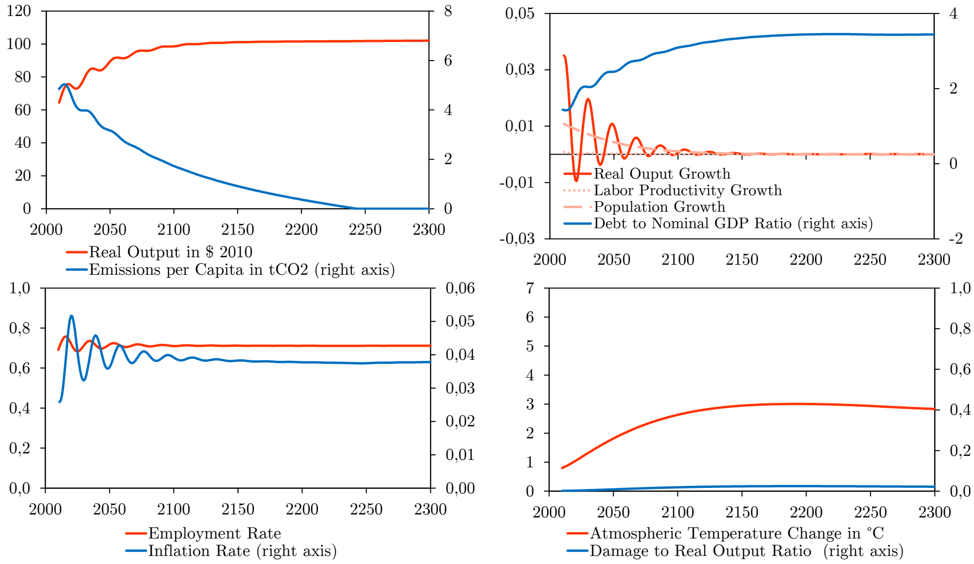

FIGURE 5.5: Trajectories in the Kaldor-Verdoorn/Nordhaus case.

Figure 5.5 depicts the path followed by the world economy in the Kaldor-Verdoorn scenario. By contrast with the previous pattern, the economy converges to a stationary state with stagnating labor productivity and no real output growth. This "millennial stagnation" starts at the end of this century and is accompanied by a debt ratio and inflation rate higher than in the exponential case. Carbon emissions decline almost immediately after 2010.[21] The zero emission floor is reached before the second half of the twenty-second century.

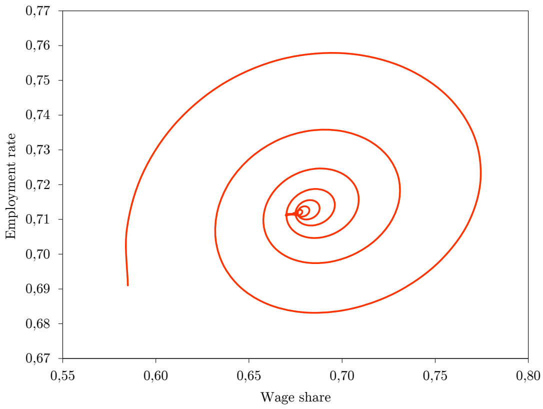

FIGURE 5.6: Phase diagram of employment rate vs. wage share in the Kaldor-Verdoorn/Nordhaus case.

Again, the almost negligible damage inflicted by global warming on the world economy, despite the fact that the average temperature change reaches approximately +2.63°C or so by the end of this century, may cast doubts on the realism of the damage function.

[21]Of course, this is not what happened —which suggests that our modelling of mitigation is too optimistic.

This thought experiment, however, reveals that, if all the parameters of the economy are kept as before and technological progress is made to depend on growth itself, this suffices, at least within our modelling approach, to break with the "infinite growth" story and leads to long-run economic stagnation.

The key variables characterizing the world situation by the end of this century are summarized in Table 5.4:

GDP Real Growth 2100 (wrt 2010)	53%
t CO_2 per capita (2050)	3.17
Temperature change in 2100	+2.63 °C
CO_2 concentration 2100	521.094 ppm

TABLE 5.4: Key values of the world economy by 2100 – the exogenous case.

5.4.2.2 The Burke *et al.* (2015) Case

This scenario allows for time-varying labor productivity that adapts endogenously to temperature anomaly. In Burke et al. (2015), a comprehensive econometric model of the dependency of world GDP growth on climate parameters is provided. In particular, a quadratic relationship between the mean annual temperature and income growth is introduced, from which we deduce the following relation between labor productivity and atmospheric temperature:

$$\frac{\dot{a}}{a} := \alpha_1 T_a + \alpha_2 T_a^2,$$

where T_a stands for the absolute atmospheric temperature and α_1, α_2 are estimated by Burke et al. (2015). Their article provides a range of seventeen models of regression.[22] We selected their most general specification[23] based on GDP growth data from the Penn World Tables in order to be in line with our own data sources. This calibration leads to $\alpha_1 \simeq 0.0072$ and $\alpha_2 \simeq -0.0004$.

Figure 5.7 shows first an increase and then, around 2150, a severe loss of labor productivity due to its nonlinear relationship with the absolute value of atmospheric temperature.

[22] Burke et al. (2015) implement a first-difference panel regression assessing a quadratic temperature impact on GDP growth with fixed effects on countries and periods, flexible country-specific trends and precipitation controls (quadratic impact). Their methodology is robust and copes with both observed and unobserved effects such as nonlinear country-specific demographic trends. They propose a range of seventeen models of regression studying several samples, an additional explanatory variable (developed and developing countries), and an alternative data source (the Penn World Tables, while their main source is the World Bank).

[23] We mean by "most general specification" the estimation realized on the full sample without an additional explanatory variable of the developed/developing country criteria.

Hence, as the change of temperature in the atmospheric layer exceeds approximately 4°C, labor productivity peaks in the region of this threshold, and real output starts decreasing. As a result, the world productive sector is forced to leverage in order to finance investment, such that the debt-to-output ratio increases twice as fast compared with the previous scenario. This failure of technological progress to fuel growth induces a delay in the private sector's deleveraging process, which results in a degrowth of real output starting in the second half of the twenty-second century. In the vicinity of the model year 2135, world real GDP peaks at around 600% of its 2010 value, and then inexorably declines. As a counterpart, the debt-to-GDP ratio explodes: it is already above 250% by 2100, and peaks slightly below 400% thereafter. Fortunately, emissions per capita have already peaked around 2050, such that the temperature change in 2100 remains lower than in the exogenous-growth scenario (+4.92°C).

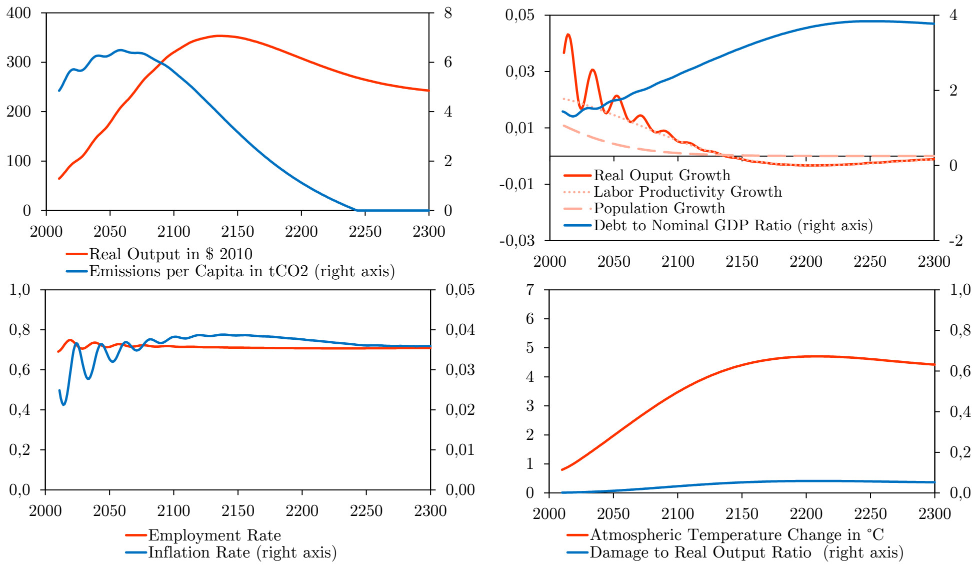

FIGURE 5.7: The Burke *et al.* labor productivity trajectory with the Nordhaus damage function.

The phase diagram in Figure 5.8 highlights the channel linking global warming, surplus distribution, and growth. Indeed, as a consequence of equation 5.10, the deceleration of labor productivity favors an increase in the wage share, lowering relative profits. This trend cuts down investment, provoking a lack of capital accumulation in the medium run, which results in a lack of ouput growth, hence of profit. This negative feedback is reinforced by the cost of too high a private debt, which again absorbs a growing part of the remaining profit. This process eventually leads to "degrowth by constraint" (not by design)

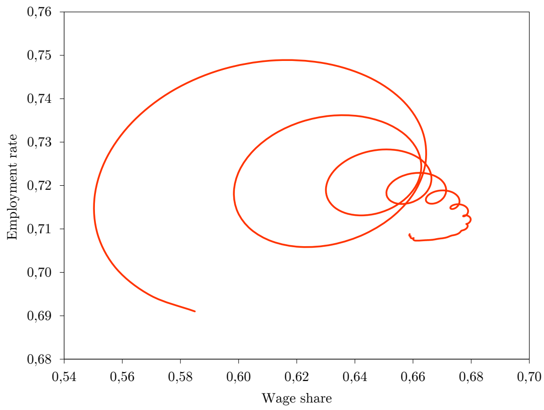

FIGURE 5.8: Phase diagram in the Burke *et al.*/Nordhaus case.

However, here, degrowth has no disruptive effect on the labor market, since the employment rate converges around 70% at the end of this century. Inflation is slightly higher than in the previous scenario but eventually converges to a quite reasonable 3.5% in the second half of the twenty-third century. Notice also that the damage function has very little impact in this scenario, as in the previous one, since the temperature anomaly peaks at around only 3.5°C or so.

GDP Real Growth 2100 (wrt 2010)	397%
t CO_2 per capita in 2050	6.29
Temperature change in 2100	+3.48 °C
CO_2 concentration in 2100	744.49 ppm

TABLE 5.5: The world economy by 2100 – the endogenous case with Nordhaus damage function.

Of course, forced degrowth at the world scale might seem an implausible pattern given the impressively innovative character of advanced economies (think of the ICT revolution) and the quite impressive growth experienced by emerging countries in recent decades. Remember, however, that during the 1990s the former Soviet Union experienced a 50% reduction of its GDP within one decade, while Greece lost 25% of its GDP between 2010 and 2015. Undesired degrowth is not therefore a fictional phenomenon.

Thus far, we have kept the damage function identical and discussed the sensitivity of our prospective paths with respect to various specifications of technological progress. Let us now proceed the other way round and test various damage functions while keeping a deterministic exponential labor productivity growth.

5.4.3 Assessing the Impact of Climate Change

So far, there is no academic consensus on a functional formulation and calibration of the damage function capturing the impact of climate on the world economy. Indeed, as pointed out by Pindyck (2015), no theory and no data exist on which such a consensus could be reliably grounded. On the other hand, this function has to summarize the economic impacts, as a percentage of output, brought on by the rise in mean atmospheric temperature. It thus has to compile a wide range of events such as biodiversity loss, ocean acidification, sea-level rise, change in ocean circulation, and high-frequency storms, among others. As a consequence, the damage function is highly nonlinear with threshold effects. In this section, various damage functions will be considered. We keep labor productivity at the somewhat high growth rate of 2.26%, so that the only difference with the BAU scenario analyzed *supra* lies in our assessment of climatic damages.

As argued by Dietz and Stern (2015), Nordhaus' quadratic form leads to unrealistically low damages beyond a temperature increase of 4°C. For instance, a global warming of 4°C would lead to only 4% of output loss whereas, according to natural science and economic studies, reaching this threshold could be a milestone. On the one hand, Lenton et al. (2008) point out that several key tipping points in the climate system could be crossed and lead to severe impacts on the natural environment. On the other hand, Stern (2013) shows that this situation would generate large migrations associated with conflict and loss of life. Our simulations in the next section will unfortunately confirm this criticism.

As a result, and in line with Pindyck (2015), we adopt the educated guess provided by Dietz and Stern (2015) with a polynomial damage function

$$\mathbf{D} = 1 - \frac{1}{1 + \pi_1 T + \pi_2 T^2 + \pi_3 T^{6.754}},$$

based on what Nordhaus proposes (same coefficients for the linear and quadratic parts). More precisely, the Weitzman function corresponds to $\pi_1 = 0$, $\pi_2 = 2.84 \times 10^{-3}$, and $\pi_3 = 5.070 \times 10^{-6}$, while the Dietz-Stern function yields $\pi_1 = 0$, $\pi_2 = 2.84 \times 10^{-3}$, and $\pi_3 = 8.19 \times 10^{-5}$.

The calibration proposed by Weitzman (2012) leads to damages equal to 50% of output with a temperature increase of 6°C. That suggested by Dietz and Stern (2015) yields damages equal to 50% of output with a temperature increase of 4°C. As emphasized by Weitzman (2012) quoting Sherwood and Huber (2010), given that a temperature increase of 12°C would exceed the human limits to metabolic heat dissipation, these educated guesses seem to be relatively credible.

Figure 5.9 presents the shapes of the different damage functions considered in this study. One can observe the common pattern for Nordhaus' and Weitzman's specifications in a global warming ranging between 0°C and 3°C.

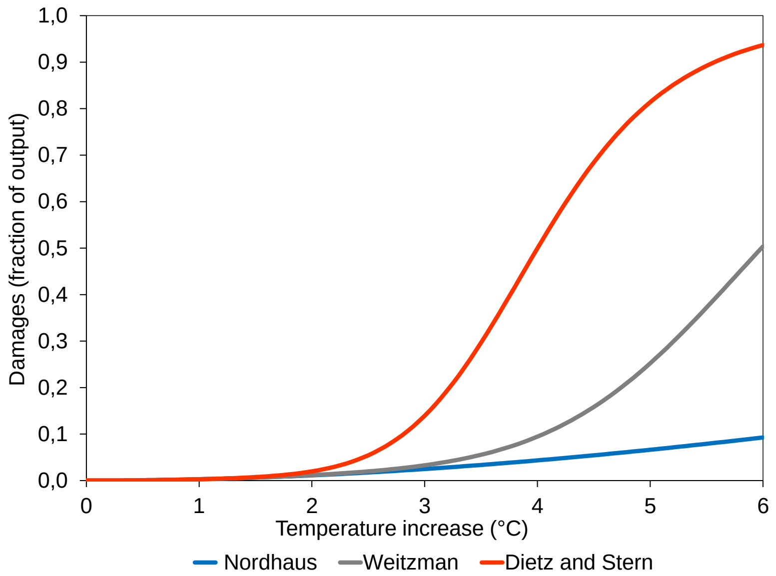

FIGURE 5.9: Shape of damage functions.

5.4.3.1 The Weitzman Damage Function

Figure 5.10 shows the trajectories of the world economy in the Weitzman damage function case. Here, the damage function drags the wage share to lower levels, leading the inflation rate towards negative values. Despite its slowdown, output growth remains positive along the trajectory. By the end of the twenty-second century, the world's real output growth rate bottoms out at -2%, then starts to increase again. As shown by the bottom right quadrant of Figure 5.10, this is clearly due to the run-up of the impact of warming with respect to production, which mainly occurs during the twenty-second century. On the other hand, the wage share, ω, stays below 35% until the end of the same century, implying high profits and the beginning of an age of deleveraging and even excess saving.[24]

[24]It is worth mentioning that the empirical estimation of investment, $\kappa(\cdot)$, as a function of the profit rate, π, is silent about domains where π has so far not been observed. This is hardly surprising: climate change will necessarily lead the world economy to explore situations for which no data can be borrowed from the past. It does however raise a question: which values should be given to investment when π is abnormally high or low? Here, we have capped and floored $\kappa(\cdot)$ between 50% and 4% of real output.

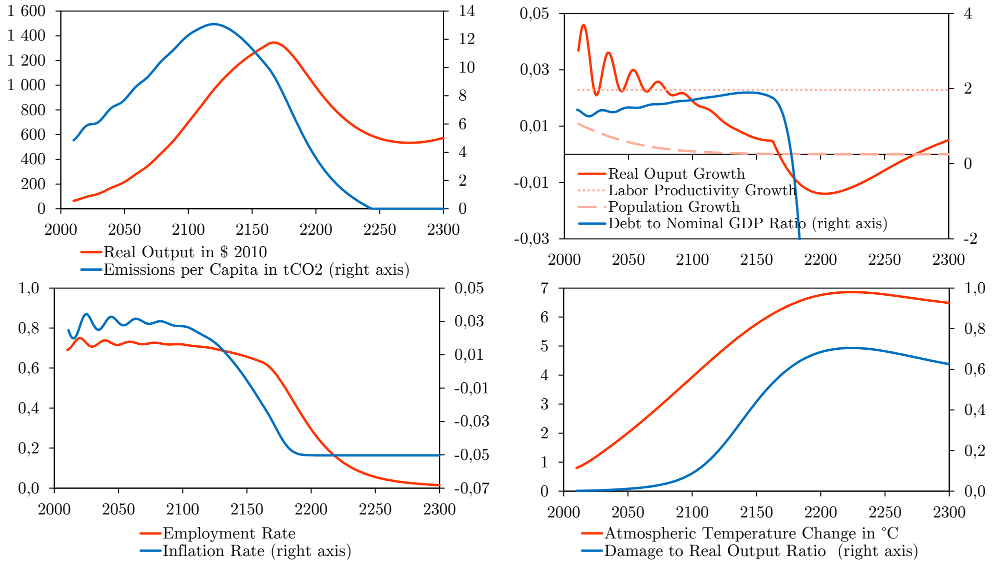

FIGURE 5.10: Trajectories of the main simulation outputs in the exponential/Weitzman case.

The phase portrait in Figure 5.11 highlights the decline of the wage share, while the employment rate steadily decreases to around 0. The fact that the world economy manages to produce some positive output growth in the second half of the twenty-third century, even though its employment rate is close to nil, confirms the unrealistic feature of our postulated exogenous growth of labor productivity.

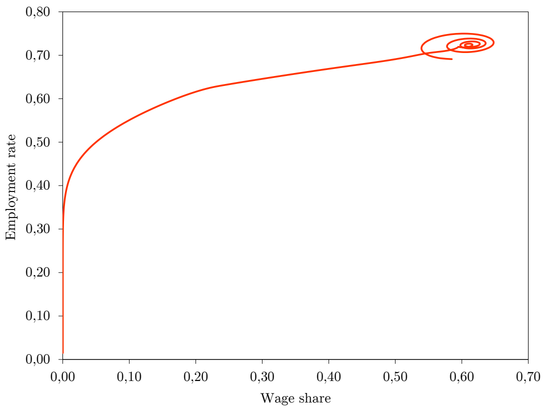

FIGURE 5.11: Phase diagram of employment rate vs. wage share in the exponential/Weitzman case.

GDP Real Growth 2100 (wrt 2010)	987%
t CO_2 per capita in 2050	7.72
Temperature change in 2100	+3.93 °C
CO_2 concentration in 2100	958.17 ppm

TABLE 5.6: The world economy by 2100 – the exogenous case with Weitzman damages.

5.4.3.2 Damages *à la* Dietz-Stern

Let us now adopt the probably more realistic Dietz-Stern damage function. Figure 5.12 shows its impact on the main macroeconomic and climate variables. Qualitatively, the short run exhibits a pattern similar to the previous scenario. Quantitatively, real GDP is more muted. In the previous scenario, it peaked in the region of (2010) US$ 1400 trillion around 2175, whereas in the current scenario the highest point is reached in 2100 at slightly above US$ 400 trillion. This more severe picture leads to a real GDP that is lower in 2175 than in 2010. In this scenario, damages absorb more than 60% of real output as the temperature increase in the upper atmosphere reaches 4°C around 2125. For the sake of comparison, at that date in the previous scenario "only" 20% of the world's real output was destroyed by global warming. Finally, the debt-to-GDP ratio spikes at around 250% towards 2125, whereas in the Weitzman case it stood below 200% for the same period. A more severe damage function reinforces the run-up to debt during the period when the economy is still growing. Note that, as previously, a deleveraging period starts whenever GDP decreases.

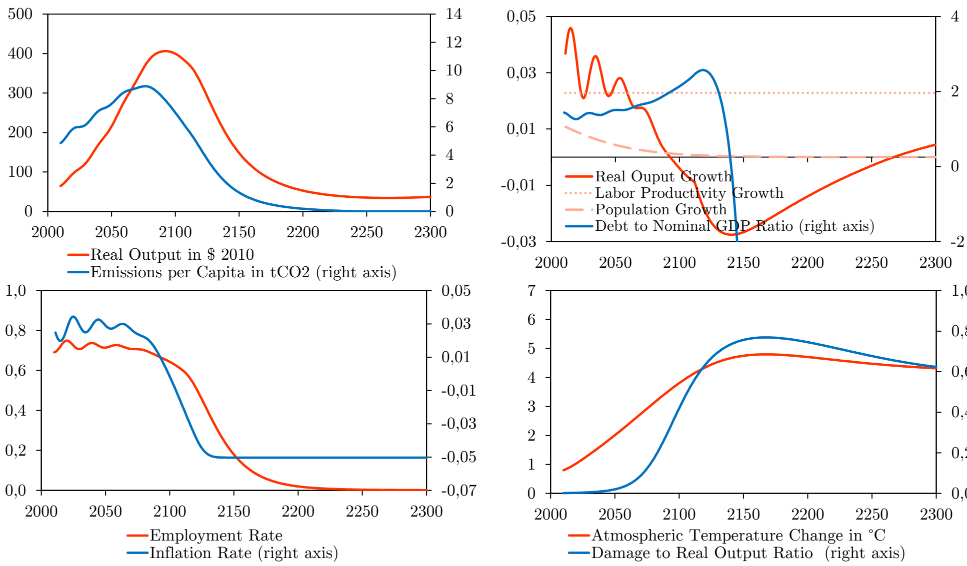

FIGURE 5.12: The exponential/Dietz-Stern case.

GDP Real Growth 2100 (wrt 2010)	520%
t CO_2 per capita in 2050	7.63
Temperature change in 2100	+3.81 °C
CO_2 concentration in 2100	857.19 ppm

TABLE 5.7: The world economy by 2100 – the exogenous case with the Dietz-Stern damage function.

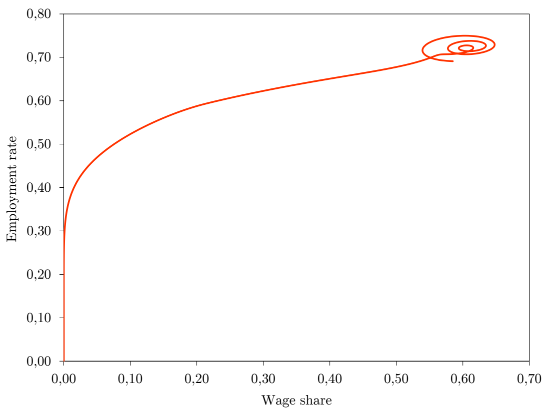

FIGURE 5.13: Phase diagram of employment rate vs. wage share in the Dietz-Stern/Nordhaus case.

5.4.4 Extreme Climate Change

In this subsection, we consider the scenarios resulting from the combination of endogenous technological progress and various damage functions.

5.4.4.1 The Burke *et al.* (2015)/Dietz-Stern case

Let us first combine endogenous labor productivity, non-linearly affected by climate change as described earlier (Burke et al. (2015)), with a highly convex Dietz-Stern damage function. This time, as shown in Figure 5.14, although the temperature increase does not exceed 4°C, the combined effects of damages and loss of labor productivity lead to a planetary collapse around 2180, preceded by severe debt-deflation. As a consequence of the depressive effect of deflation and the subsequent breakdown of the world economy, the peak of CO_2 emissions is lower than in most of the previous scenarios, and occurs around 2080. Yet, due to the inertia effects of global warming, this early peaking cannot prevent a planetary collapse one century later.

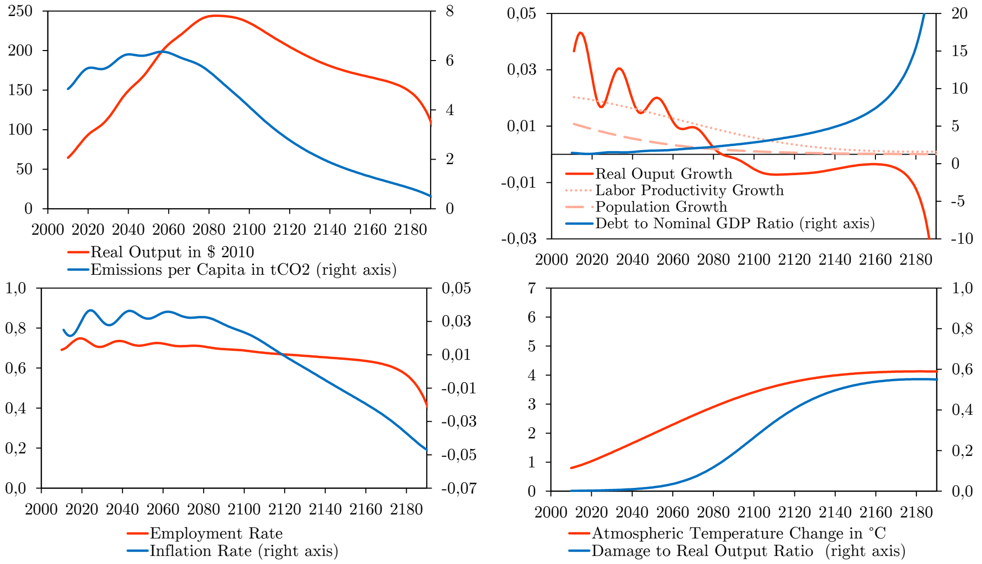

FIGURE 5.14: The Burke *et al.* (2015)/Dietz-Stern case.

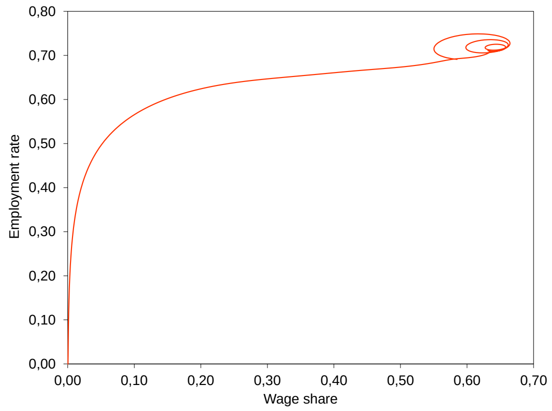

FIGURE 5.15: Phase diagram of employment rate vs. wage share in the Burke *et al.*/Dietz-Stern case.

5.4.4.2 The Burke *et al.* (2015)/Dietz-Stern case with a Slower Demographic Trend

Could a deceleration of the demographic trend prevent a disaster? This subsection provides some elements for an answer by assuming the demographic trend to be slower than in the UN median scenario within the Burke *et al.* labor productivity growth case, together with the Dietz-Stern damage function previously presented. For this purpose, we divide by four the speed of convergence, q, but we keep the same upper bound for the dynamics of the labor force, M. Figure 5.16 offers a comparison of the demographic scenarios.

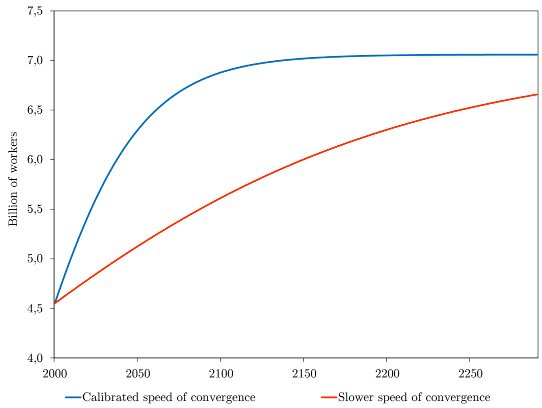

FIGURE 5.16: Comparison of the labor-force demographic trajectories.

According to our altered demographic scenario, the world's working-age population would be approximately 5 billion people in 2100, instead of 7 billion as in the UN median projection. Figure 5.17 shows the paths followed by the world economy in this case. Despite lower CO_2 emissions, we observe patterns analogous to those obtained in the original Burke/Dietz-Stern case, leading to a global collapse around the model year 2240. The main difference between the two narratives lies in the speed at which events occur: the second narrative exhibits a 4–5-decade delay relative to the first. This suggests that a downturn in the demographic trend does not suffice *per se* to avoid a disaster, but it nevertheless manages to postpone it for a few decades.

Unfortunately, other simulations, even with no population growth,[25] show that in the Burke/Dietz-Stern case a global collapse *always* occurs whatever the population trend. Even in the utterly unrealistic case in which world population stays at its 2010 level, the intrinsic devastating forces arising from the combination of climate change and debt would lead to a breakdown around 2400. In terms of public policy, this means that steering world population growth cannot be viewed as a panacea, but it *does* have a positive impact on the global economic calendar of our planet.

[25]These simulations are available from the authors upon request.

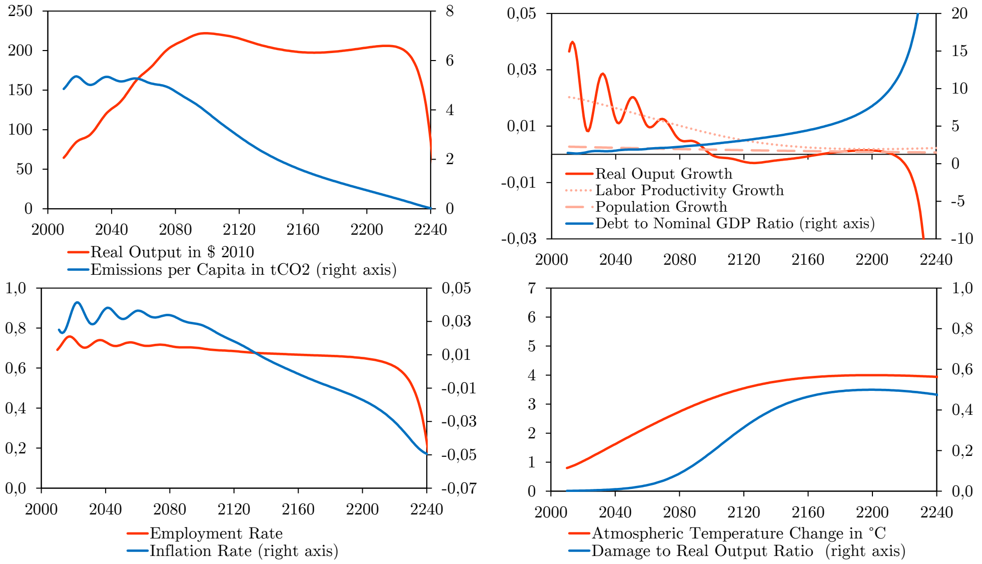

FIGURE 5.17: Trajectories of the main simulation outputs for the case with the Burke *et al.* (2015) labor productivity growth, a Dietz-Stern damage function, and a slower demographic trend.

5.4.5 Carbon Prices and Climate Sensitivity

Since demography alone does not suffice to circumvent the potentially disastrous effects of global warming, we now turn to the carbon value. So far, we have considered the baseline scenario of the carbon price introduced by Nordhaus and Sztorc (2013). For the sake of clarity, the price of the t/CO_2 in (2005) \$US is one in 2010 and grows steadily by two percent per year.

In this section, we retain the Burke et al. (2015) labor productivity dynamic coupled with a Dietz-Stern damage function, but modify the carbon price path, taking inspiration from Dietz and Stern (2015). On the demographic side, we again adopt the UN median projection, as we do throughout this paper except for Section 5.4.4.2 above.

5.4.5.1 Dietz and Stern's Standard-run Price

We now assume that the carbon price follows the path examined in Dietz and Stern (2015), starting with (2005) \$US 12 t/CO_2 in 2015 and reaching \$US 29 t/CO_2 in 2055.

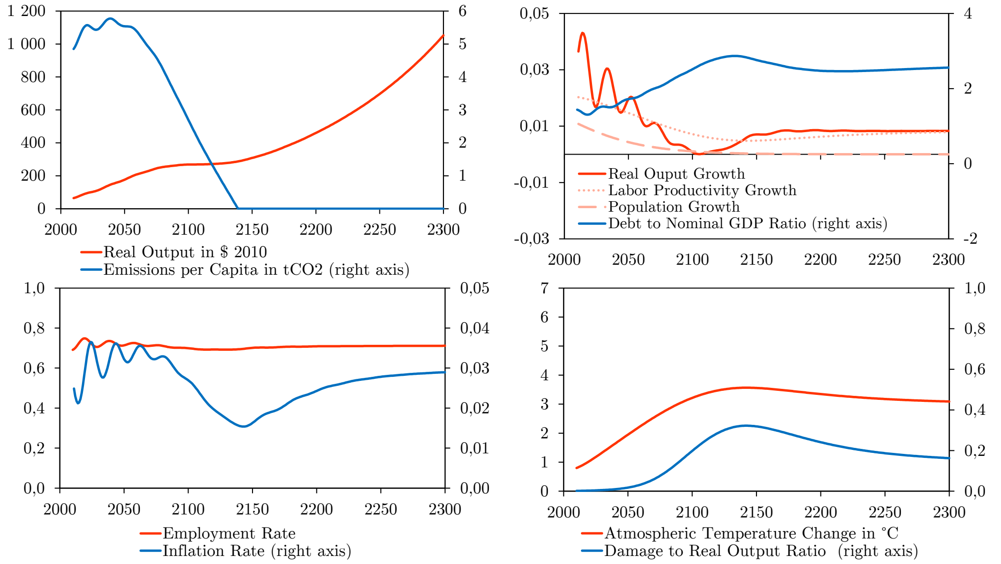

FIGURE 5.18: Trajectories in the Burke *et al.* (2015)/Dietz-Stern case with Stern's standard-run carbon price.

Figure 5.18 shows that Stern's carbon price path suffices to avoid the collapse. The higher carbon price prevents the +4°C temperature anomaly from being reached. As a result, damages are mitigated and, courtesy of the wage share dynamic, private debt remains at a reasonable level (less than 300%).

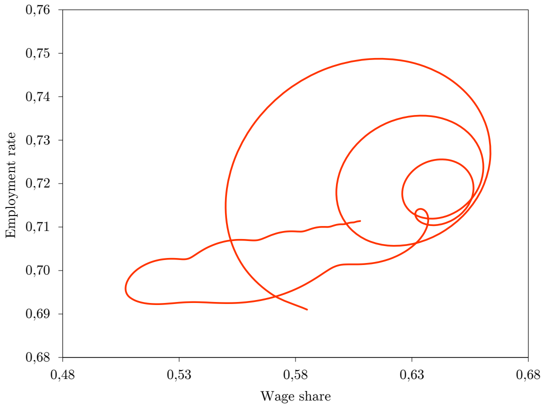

FIGURE 5.19: Phase diagram of employment rate vs. wage share in the Burke *et al.*/Dietz-Stern case with Stern's carbon price path.

Figure 5.19 shows how the (ω, λ)-point converges towards long-run equilibrium. The same intriguing phenomenon as in Figure 3 *supra* is to be observed: climate change succeeds in temporarily driving the world economy away from its long-run stationary state, but fortunately does not seem to be strong enough to push it across the "potential barrier" surrounding the "good" equilibrium. By contrast with the last scenario, Stern's carbon price suffices in elevating the height of the potential barrier so as to protect the world economy – not every carbon price path successfully accomplishes this task, however. Moreover, this success is clearly due to the utterly simple, and probably unrealistic, way we model the world economy's shift from a current energy mix comprising 80% of fossil energies towards a zero-carbon economy.

5.4.5.2 Dietz and Stern's Standard-run Price with a Climate Sensitivity of 6

So far, all our results are based on a climate sensitivity whereby a doubling in the atmospheric concentration of CO_2 translates into a +2.9°C rise relative to the pre-industrial era. This value reflects the mean of a Pareto distribution whose tail yields a 6% likelihood that a rise of +6°C or more will occur in these circumstances (see Weitzman (2011)). We thus test some of our scenarios under a climate sensitivity of 6, rather than 2.9.[26] Clearly, in this setting, any variation of CO_2 will lead to a higher response in temperature anomaly compared to its +2.9°C counterpart.

We begin with the scenario whose *terminus ad quem* examined thus far is the most problematic, namely that in subsection 5.4.4.1 *supra*, which combines an endogenous labor productivity *à la* Burke *et al.*(2015) and a highly convex Dietz-Stern damage function. At variance with the situation envisaged in the subsection just mentioned, here, we keep the carbon price path introduced by Stern (instead of relying on Nordhaus's price path as in section 5.4.4.1).

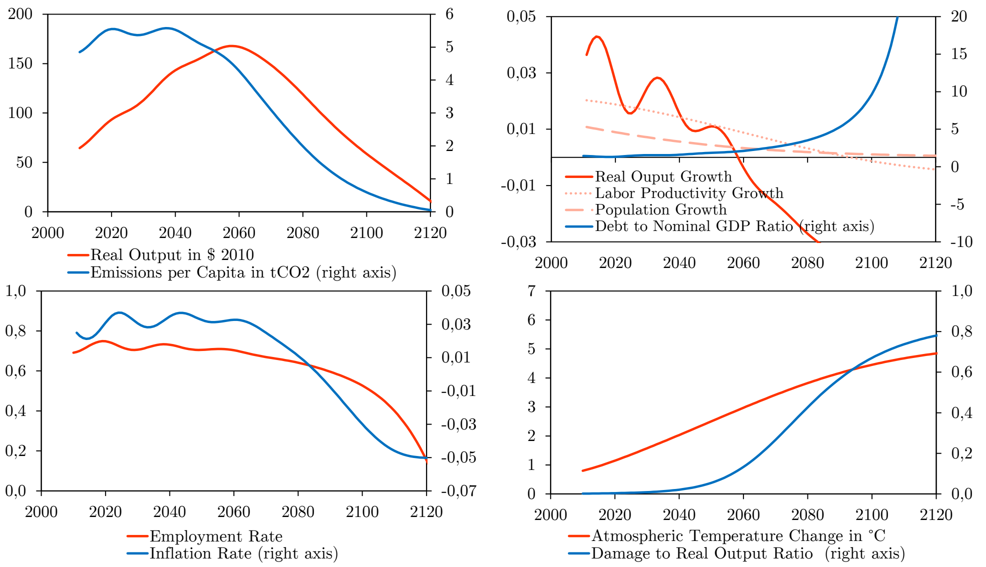

FIGURE 5.20: Trajectories in the Burke *et al.* (2015)/Dietz-Stern case with the standard-run price of carbon and a climate sensitivity of 6.

[26]The recent contribution of Snyder (2016) suggests that the climate sensitivity could be higher than current estimates. To stay in line with the consensual position of the Fifth Assessment of the Stocker et al. (2013), we only consider a maximum climate sensitivity of 6 for the purposes of this paper. However, the possibility of an even more severe global warming induced by CO_2 accumulation should not be excluded for future analysis.

Figure 5.20 shows that the preceding specification of the carbon price no longer avoids a collapse of the economy. The cap of a +4°C temperature rise relative to the pre-industrial level is reached long before 2100. Consequently, high damages together with the inertia of CO_2 in the atmospheric layer lead the world economy to deflation and a skyrocketing debt ratio, yet again ending up in a global breakdown.

Next, we test the carbon price path more recently introduced by Dietz and Stern (2015) for a climate sensitivity of 6 and a damage function *à la* Dietz-Stern. Converted into 2005 $US, in 2015 the price of the ton of CO_2 is now US$ 74, and US$ 306 in 2055.

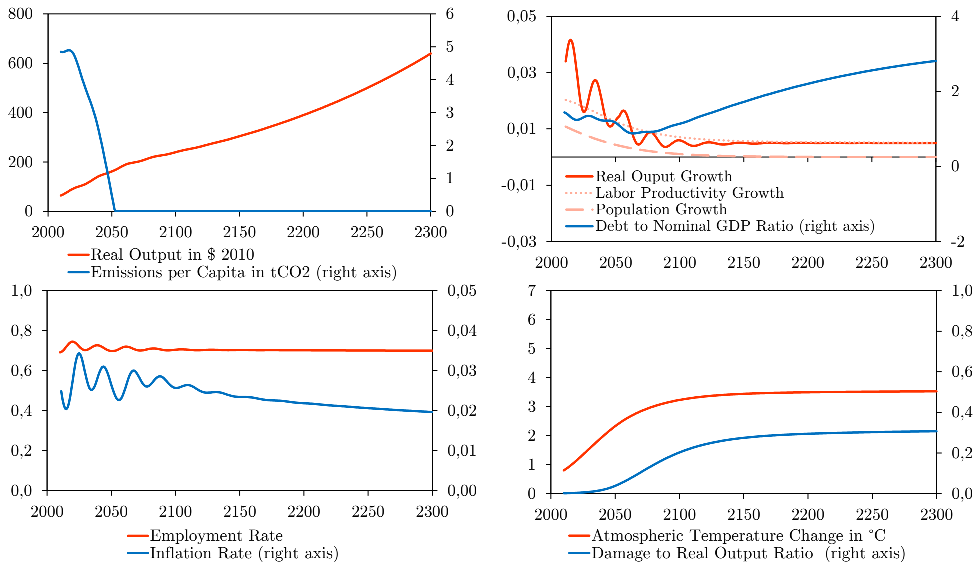

FIGURE 5.21: Trajectories in the Burke *et al.* (2015)/Dietz-Stern case,the Dietz-Stern carbon price path, and a climate sensitivity of 6.

This time, the carbon price path turns out to be sufficient to avoid the collapse. Figure 5.21 displays a trajectory in which real GDP in 2100 reaches about 2.72 times its value in 2010, with the emission of t CO_2 per capita decreasing to 0.70 in 2050 and the temperature increasing to only +3.23°C in 2100.

5.4.5.3 Objective +1.5°C

The Paris Agreement of 2015 aims to keep the temperature anomaly below +2°C and drive efforts to stay as close as possible to a +1.5°C threshold. Is such a target reachable according to the framework developed in this paper?

Again, we base our analysis on the scenario whose conclusion is the most worrisome, namely the Burke et al. (2015) labor productivity growth together with a Dietz and Stern

(2015) damage function. Within this framework, the doubling in CO_2 concentration implies a temperature anomaly of 6°C (i.e., if the climate sensitivity is equal to 6), while the increase in temperature turns out to be already about +1.52°C in 2100 even if CO_2 emissions were to remain at their 2010 level. Thus, in our set-up, there is no hope of reaching the 1.5°C target were the climate sensitivity equal to 6.

We therefore consider lower climate sensitivities ranging from 1.5 to 2.9. In this setting, we test which carbon price path could prevent the temperature anomaly from exceeding the +1.5°C ceiling. For this, we use a constant initial condition for the carbon price in 2010 (either US$ 15, or US$ 80) and look for the per annum growth parameter that prevents the temperature anomaly from exceeding +1.5°C in 2100.[27]

	Sensitivity of +1.5°C		**Sensitivity of +2.9°C**	
	Init. price of 15	Init. price of 80	Init. price of 15	Init. price of 80
Price in 2015	18.58	86.27	65.50	144.32
Price in 2020	23.00	93.04	286.02	260.35
Price in 2050	82.93	146.35	*xxx*	*xxx*

TABLE 5.8: Carbon prices preventing the temperature anomaly from reaching the 1.5°C ceiling, in (2005) US$/$tCO_2$.

Table 5.8 provides some carbon price paths that prevent the temperature anomaly from exceeding the 1.5°C ceiling. When *xxx* is reported, the simulated values are higher than 2005 US$ 344 (the maximum price of the backstop technology, in the same currency unit, needed to complete the energy shift), and thus meaningless. As expected, prices need to be higher in the 2.9 case than in the 1.5 case in order to reach the 1.5°C target.

The 1.5°C sensitivity case shows that the necessary carbon price increase is lower when the initial price is higher.[28] The price of CO_2 must reach US$ 300 in 2080 and 2100 respectively, for an initial price of US$ 15 and US$ 80. This means that the energy shift should be completed around the end of the twenty-first century.

In the 2.9°C sensitivity case, despite different starting values, the value of the carbon price in 2020 must be higher than US$ 260 per ton of CO_2. In our set-up, this implies

[27] We do not claim that the values obtained here are minimal (whatever the sense one might wish to give this here) in order to reach the +1.5°C target. At best, they are educated guesses given our numerical experience with the present model. We believe, however, that they provide a faithful indication of what a more systematic exploration of this model's sensitivity relative to the carbon price path would provide. The latter is left for further research.

[28] This reflects the ongoing debate between Stern and Nordhaus, the former advocating a high starting point with a low increase in the subsequent decades, while the latter defends a low initial condition, followed by a sharper carbon price increase.

that the energy shift should be almost completed by 2020. Needless to say, by the time this paper is written, there is little hope that the world economy will reach zero carbon emissions by 2020. Thus, we view this last result as indicating that as soon as the climate sensitivity is 2.9, it is already too late to reach the +1.5°C target.

5.5 Conclusion and Directions for Further Work

By combining financial and environmental aspects, the stock-flow consistent macroeconomic model introduced in this paper allows us to evaluate economic growth, or possible (forced) degrowth, depending on the dynamics of labor productivity, damages induced by global warming, the demographic trend, and climate sensitivity, as well as the carbon price path. To our knowledge, this is the first dynamic model estimated at world level that enables both environmental and financial risks to be assessed within a framework of endogenous monetary business cycles.

Our main findings are the following: when a relatively realistic growth path for technological progress is adopted, taking due account of the influence of global warming on labor productivity, a quite (i.e., significantly convex) reasonable damage function leads to a possible breakdown of planetary magnitude either before or around the next century's turning point. Second, curbing the demographic trend does indeed postpone the potential disaster but is not sufficient to avoid it. Third, a carbon price starting at US\$ 12 t/CO_2 in 2015 and reaching US\$ 29 t/CO_2 in 2055 suffices to restore perpetual growth whenever climate sensitivity is 2.9. With a high climate sensitivity of 6, a much more severe carbon price path is needed, starting for instance at US\$ 65.5 t/CO_2 in 2015 and finishing at a level higher than US\$ 285 t/CO_2 in 2050. Given the radical uncertainty that plagues climatologists' knowledge about climate sensitivity, these results call for strong and immediate action. This can take the form of a high carbon price (or price corridor, since there is no reason for the relevant incentivizing price to be uniform throughout the world), starting immediately above US\$ 65.5 t/CO_2, and rapidly increasing. Finally, it seems too late for the world economy to be able to reach the +1.5°C target, unless with a stroke of luck climate sensitivity turns out to be very low (1.5).

These results complete the path-breaking work of LtG by adding a third cause of possible collapse to the scarcity of natural resources and pollution (other than CO_2 emissions). It also *a posteriori* justifies our choice not to follow a standard cost-benefit analysis to assess the impact of climate-driven externalities. Certainly, the latter approach inevitably ends up with the issue of calibrating the "right" discount rate. While substantial efforts have been devoted to assessing whether a high or low, and sometimes a time-varying, discount rate should be considered,[29] none of this literature, to the best of our knowledge, has ever considered a negative rate.[30] Yet, this possibility should be seriously envisaged. Not only because of the pervasive negative interest rates observed nowadays on international markets, but also, as shown in this paper, because a world breakdown might be the prospect that markets should start facing from now on. If the next generation is going to be less wealthy than we are today, then a US dollar today should be worth less than the same dollar in a couple of decades.

However striking our findings may be, we view them as only a preliminary step that points to a number of areas to be deepened, among which the following seem prominent:

1. In a subsequent work, we plan to couple the climate feedback loop introduced here with the modeling of non-renewable natural resource scarcity. It is only to be expected that this additional reality will cause the possible collapse to emerge sooner and more severely. It will also enable us to envision more realistic answers to be provided by the international community.

2. Here, we contented ourselves with studying business-as-usual scenarios. Our rather pessimistic conclusions by no means imply that we believe a planetary breakdown is unavoidable. First, in this paper, we have shown that certain carbon price paths can provide the appropriate incentives for a fast shift towards clean energy. This, of course, needs to be qualified considering the rather simplistic way we captured such a shift. Further work will be necessary to take due account of some of the difficulties involved in directed technological change. In particular, the problem of hysteresis due to the *ex post* non-substitutability of capital seems to us to be a key issue. The following example aptly illustrates our point: the world economy today counts approximately one billion combustion-engine vehicles, none of which can be easily converted to gas- or electric-powered vehicles. Even though a high carbon value might provide a strong incentive to develop a green mobility market, the question would still remain as to what households and industry should do with already existing cars. More generally, understanding how a world disaster may be avoided requires introducing the public sector. Public policy, however, is also constrained by public finances. The dynamic of taxes, public spending, and public debt will presumably be key in building realistic paths that can successfully circumvent the breakdown.

[29] see, e.g., Sterner and Persson (2008)

[30] Except for Ivar Ekeland who introduced an "ecological interest" rate in Ekeland (2015), p.49. For Ekeland, consumption goods (available in large quantities) and natural resources (available in limited quantities) should be valued using two different interest rates. While the first one can be set by the market, the second one should be lower or negative due to its finiteness.

3. Even though its dynamic is already quite rich and its empirical calibration probably as accurate as possible (given current data availability constraints), the macroeconomic set-up introduced here provides but a stylized framework. We view it more as a proof of concept than a precise, prospective picture. More empirical accuracy will be gained by adding some of the following additional features:

   - Introducing some (medium-run) substitutability between capital and labor (e.g., along the lines of Grasselli and Maheshwari (2016)) will give the model a closer fit with our daily experience, and help us understand how the productive sector might actually react more effectively to the challenges raised by global warming, as it has been assumed in the present paper;

   - Making explicit the allocation of capital, equity markets, and the banking sector (following, e.g., Giraud and Kockerols (2015)) should make it possible to study in greater depth the banking sector's capacity to fund investment;

   - Dropping Say's law by decoupling supply and demand (as is the case in Giraud and Grasselli (2016) or Grasselli and Nguyen Huu (2016)) will make it possible to understand how a more sober consumption pattern may help circumvent a disaster. On the other hand, from a Keynesian perspective, it will also enable a study of how deflation impacts the fall in demand;

   - Adding damage costs to current investment (instead of subtracting them) will enable to study the relevance of the "Jevons paradox" or "rebound effect" at a macro level, as emphasized by Rezai et al. (2013).

   - In this paper, firms are supposed to behave myopically in the sense that aggregate investment is empirically estimated as a function of current profit. Adding expectations (e.g., adaptive ones) would allow the robustness of our findings to be checked with respect to more sophisticated behaviors;

   - Here, investment and the short-run Phillips curve are estimated using some linear OLS methods with Gaussian residuals (see Appendix D.5). The robustness of our findings will be checked in a companion paper, using a polynomial, non-Gaussian estimation of these aggregate behaviors.

4. In LtG, agricultural production was distinguished from industrial output, and a number of scenarios indicated that the former would decline before the latter. Understanding this timing is also crucial in order to design efficient public policies.

A next step would therefore be to extend the present framework to a multisectoral macrodynamics with heterogeneous types of capital and consumption commodities.

5. As we saw above, the precise determination of the damage functions plays a crucial role in assessing the possibility of a breakdown. Even though they are borrowed from the literature, the functions employed here deserve a more careful definition. We plan to rewrite them by quantifying the economic impact of the rise in sea level, glacier melting, soil erosion, etc. This not only requires a multisectoral standpoint (see 4. above) but also a geographical disaggregation of the broad planetary perspective adopted here.

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Appendix A

Appendix Chapter 2

A.1 Model Derivations

Household

The problem of each household is:

$$\begin{aligned} \max_{\{C_t, L_t, B_t, K_{t+1}\}} \quad & \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \big[U(C_t, L_t)\big], \quad 0 < \beta < 1, \\ \text{subject to :} \quad & P_{e,t}C_{e,t} + P_{q,t}C_{q,t} + P_{k,t}(K_{t+1} - (1-\delta)K_t) + B_t \\ \leq \quad & (1 + i_{t-1})B_{t-1} + W_t L_t + D_t + r_t^k P_{k,t} K_t + T_t, \end{aligned}$$

where the consumption flow is defined as:

$$C_t := \Theta_x C_{e,t}^x C_{q,t}^{1-x}, \tag{A.1}$$

with $x \in (0,1)$ being, at equilibrium, the share of oil in consumption, $\Theta_x := x^{-x}(1-x)^{-(1-x)}$, and $C_{q,t} := \left(\int\limits_{[0,1]} C_{q,t}{}^{\frac{\epsilon-1}{\epsilon}}(i)di \right)^{\frac{\epsilon}{\epsilon-1}}$ is a CES index of domestic goods. Note that, from (A.1), a fraction of imported oil is consumed by households.

In order to ensure that this programme has a solution, we impose the following transversality condition (no Ponzi scheme):

$$\lim_{k\to\infty} \mathbb{E}_t \left(\frac{B_{t+k}}{\prod_{s=0}^{t+k-1} (1+i_{s-1})} \right) \geq 0, \quad \forall t.$$

The optimal allocation of expenditures among different goods, domestic and foreign, yields:

$$\begin{aligned} P_{q,t}C_{q,t} &= (1-x)P_{c,t}C_t \\ P_{e,t}C_{e,t} &= xP_{c,t}C_t \\ \text{CPI index:} \quad P_{c,t} &= P_{e,t}^x P_{q,t}^{1-x} \end{aligned}$$

The Lagrangian associated with the maximization problem of the household has the following form:

$$\begin{aligned} \mathcal{L}_0 &= \sum_{t=0}^{\infty} \beta^t \mathbb{E}_0 \Big[U(C_t, L_t) - \lambda_t \big[P_{c,t}C_t + P_{k,t}I_t \\ &\quad + B_t + T_t + (1+i_{t-1})B_{t-1} + W_t L_t + D_t + r_t^k P_{k,t} K_t \big] \Big] \end{aligned}$$

Where λ_t is the Lagrange multiplier. The first order conditions are:

$$\begin{aligned} C_t: &\quad U_C(C_t, L_t) = \lambda_t P_{c,t} \\ L_t: &\quad U_L(C_t, L_t) = \lambda_t W_t \\ B_t: &\quad \lambda_t = \beta \mathbb{E}_t \Big[(1+i_t)\lambda_{t+1} \Big] \\ K_{t+1}: &\quad \lambda_t P_{k,t} = \beta \mathbb{E}_t \Big[\lambda_{t+1} \big(r_{t+1}^k + 1 - \delta \big) P_{k,t+1} \Big]. \end{aligned}$$

Therefore, we have the following inter-temporal optimal conditions:

$$\begin{aligned} 1 &= \beta \mathbb{E}_t \Big[(1+i_t) \frac{C_t}{C_{t+1}} \frac{P_{c,t}}{P_{c,t+1}} \Big] \\ 1 &= \beta \mathbb{E}_t \Big[\frac{C_t}{C_{t+1}} \frac{P_{c,t}}{P_{c,t+1}} \frac{P_{k,t+1}}{P_{k,t}} (r_{t+1}^k + 1 - \delta) \Big] \\ \frac{W_t}{P_{c,t}} &= C_t L_t^{\phi} \end{aligned}$$

One can define:

1. The stochastic discount factor from date t to date $t+1$ by:

$$d_{t,t+1} := \frac{\beta U_C(C_{t+1}, L_{t+1})}{U_C(C_t, L_t)} \frac{P_{c,t}}{P_{c,t+1}} =: \Delta_t^{t+1}, i.e, \quad \frac{1}{1+i_t} = \mathbb{E}_t(d_{t,t+1}).$$

2. The stochastic discount factor from date t to date $t+k$ by:

$$d_{t,t+k} := \prod_{s=t}^{t+k-1} \Delta_s^{s+1}, \text{ then, } \quad d_{t,t+k} := \frac{\beta^k U_C(C_{t+k}, L_{t+k})}{U_C(C_t, L_t)} \frac{P_{c,t}}{P_{c,t+k}}.$$

Final Good Firm

A representative final good firm maximizes its profit without market power.

$$\begin{aligned} & \max_{Q_t(\cdot)} P_{q,t} Q_t - \int_{[0,1]} P_{q,t}(i) Q_t(i) di \\ \text{subject to } : \quad & Q_t = \Big(\int_{[0,1]} Q_t(i)^{\frac{\epsilon-1}{\epsilon}} di \Big)^{\frac{\epsilon}{\epsilon-1}} \end{aligned}$$

The first order condition for $Q_t(i)$ is:

$$\begin{aligned} P_{q,t} \frac{\epsilon}{\epsilon-1} \Big(\int_{[0,1]} Q_t(i)^{\frac{\epsilon-1}{\epsilon}} di \Big)^{\frac{\epsilon}{\epsilon-1}-1} & \quad \frac{\epsilon-1}{\epsilon} Q_t(i)^{\frac{\epsilon-1}{\epsilon}-1} - P_{q,t}(i) = 0 \\ Q_t(i) & = \left(\frac{P_{q,t}(i)}{P_{q,t}} \right)^{-\epsilon} Q_t \end{aligned}$$

The price of the final good will therefore be:

$$P_{q,t} = \Big(\int_{[0,1]} P_{q,t}(i)^{1-\epsilon} di \Big)^{\frac{1}{1-\epsilon}}. \tag{A.2}$$

Intermediate Goods Firms

One can assume the following production function for the intermediate good firm i:

$$\begin{aligned} Q_t(i) & := A_t E_t(i)^{\alpha_e} L_t(i)^{\alpha_\ell} K_t(i)^{\alpha_k} \\ & \alpha_e, \alpha_\ell, \alpha_k \geq 0. \end{aligned}$$

Intermediate goods firms solve a two-stage problem. Firstly, the costs minimization and, secondly, the profit maximization.

Costs minimization

The Lagrangian associated to the problem is:

$$\mathcal{L}_0 = P_{e,t}E_t(i) + W_tL_t(i) + r_t^kP_{k,t}K_t(i) - mc_t(i)\Big(A_tE_t(i)^{\alpha_e}L_t(i)^{\alpha_\ell}K_t(i)^{\alpha_k} - Q_t(i)\Big)$$

The first order conditions are:

$$\begin{aligned}
E_t(i): &\qquad P_{e,t} = mc_t(i)\alpha_e AE_t(i)^{\alpha_e-1}L_t(i)^{\alpha_\ell}K_t(i)^{\alpha_k}\\
L_t(i): &\qquad W_t = mc_t(i)\alpha_\ell AE_t(i)^{\alpha_e}L_t(i)^{\alpha_\ell-1}K_t(i)^{\alpha_k}\\
K_t(i): &\qquad r_t^kP_{k,t} = mc_t(i)\alpha_k AE_t(i)^{\alpha_e}L_t(i)^{\alpha_\ell}K_t(i)^{\alpha_k-1}.
\end{aligned}$$

Hence, the following relation must hold:

$$\frac{W_tL_t(i)}{\alpha_\ell} = \frac{r_t^kP_{k,t}K_t(i)}{\alpha_k} = \frac{P_{e,t}E_t(i)}{\alpha_e}.$$

On the other hand, we have:

$$\begin{aligned}
Q_t(i) &= A_tE_t(i)^{\alpha_e}L_t(i)^{\alpha_\ell}K_t(i)^{\alpha_k}\\
&= A_t\big(\frac{\alpha_e mc_t(i)Q_t(i)}{P_{e,t}}\big)^{\alpha_e}\big(\frac{\alpha_\ell mc_t(i)Q_t(i)}{W_t}\big)^{\alpha_\ell}\big(\frac{\alpha_k mc_t(i)Q_t(i)}{r_t^kP_{k,t}}\big)^{\alpha_k}\\
&= \frac{A_t\alpha_e^{\alpha_e}\alpha_\ell^{\alpha_\ell}\alpha_k^{\alpha_k}}{P_{e,t}^{\alpha_e}W_t^{\alpha_\ell}(r_t^kP_{k,t})^{\alpha_k}}\big[mc_t(i)Q_t(i)\big]^{\alpha}.
\end{aligned}$$

Where $\alpha := \alpha_e + \alpha_k + \alpha_l$. Defining $F_t := \Big(\frac{A_t\alpha_e^{\alpha_e}\alpha_\ell^{\alpha_\ell}\alpha_k^{\alpha_k}}{P_{e,t}^{\alpha_e}W_t^{\alpha_\ell}(r_t^kP_{k,t})^{\alpha_k}}\Big)^{\frac{-1}{\alpha}}$.

Thus,

$$mc_t(i) = F_tQ_t(i)^{\frac{1}{\alpha}-1}$$

And the cost function is:

$$cost(Q_t(i)) = \alpha F_tQ_t(i)^{\frac{1}{\alpha}}$$

Profit Maximization under Flexible Price

At each date t, firm i's profit maximization problem is:

$$\max_{P_{q,t}(i)} P_{q,t}(i)Q_t(i) - cost(Q_t(i))$$
$$\text{subject to} \qquad Q_t(i) = \Big(\frac{P_{q,t}(i)}{P_{q,t}}\Big)^{-\epsilon} Q_t.$$

Note that this problem does not depend on i. Consequently, its solution $P_{q,t}(i)$ does not depend on i, i.e. $P_{q,t}(i) = P^o_{q,t}$ for every i. Combining with (A.2), we have $P_{q,t}(i) = P_{q,t}$ for every i.

The first order condition for $P^o_{q,t}$ gives:

$$P^o_{q,t} = \frac{\epsilon}{\epsilon - 1} mc^o_t,$$

where $mc^o_t := F_t Q_t^{\frac{1}{\alpha}-1}$.

Profit Maximization under Calvo Price setting

In each period, the firm i has a probability θ to not reset its price. At each date t, firm i's profit maximization problem is:

$$\max_{P_{q,t}(i)} \mathbb{E}_t\Big[\sum_{k=0}^{\infty} \theta^k d_{t,t+k}\big[P_{q,t}(i)Q_{t,t+k}(i) - cost(Q_{t,t+k}(i))\big]\Big]$$
$$\text{subject to} \qquad Q_{t,t+k}(i) = \Big(\frac{P_{q,t}(i)}{P_{q,t+k}}\Big)^{-\epsilon} Q_{t+k}, \quad \forall k \geq 0.$$

Note that this problem does not depend on i, hence its solution $P_{q,t}(i)$ does not either, we write: $P_{q,t}(i) = P^o_{q,t}$. The first order condition for $P^o_{q,t}$ is:

$$\mathbb{E}_t \sum_{k=0}^{\infty} \theta^k d_{t,t+k} Q^o_{t,t+k}\Big[P^o_{q,t} - \mathcal{M}_p mc^o_{t,t+k}\Big] = 0,$$

where: $\mathcal{M}_p := \frac{\epsilon}{\epsilon-1}$, $mc^o_{t,t+k} := F_{t+k}(Q^o_{t,t+k})^{\frac{1}{\alpha}-1}$, and $Q^o_{t,t+k} := \Big(\frac{P^o_{q,t}}{P_{q,t+k}}\Big)^{-\epsilon} Q_{t+k}$ for every $k \geq 0$.

The next three lemmas show the integration of the production function using Calvo price setting.

Lemma A.1. *The aggregate production function is:*

$$\Big(\int_{[0,1]} (\frac{P_{q,t}(i)}{P_{q,t}})^{\frac{-\epsilon}{\alpha}} di\Big)^{\alpha} Q_t = A_t E_t^{\alpha_e} L_t^{\alpha_\ell} K_t^{\alpha_k}.$$

Proof. One has

$$\begin{aligned}\Big(\frac{P_{q,t}(i)}{P_{q,t}}\Big)^{-\epsilon} Q_t &= Q_t(i) = A_t E_t(i)^{\alpha_e} L_t(i)^{\alpha_\ell} K_t(i)^{\alpha_k} \\ &= A E_t(i)^{\alpha_e} \Big(\frac{P_{e,t}E_t(i)}{W_t}\frac{\alpha_\ell}{\alpha_e}\Big)^{\alpha_\ell} \frac{P_{e,t}E_t(i)}{r_t^k P_{k,t}} \frac{\alpha_k}{\alpha_e}^{\alpha_k} \\ &= A_t E_t(i)^{\alpha} \Big(\frac{P_{e,t}}{W_t}\frac{\alpha_\ell}{\alpha_e}\Big)^{\alpha_\ell} \Big(\frac{P_{e,t}}{r_t^k P_{k,t}}\frac{\alpha_k}{\alpha_e}\Big)^{\alpha_k}.\end{aligned}$$

Hence we get

$$\Big(\frac{P_{q,t}(i)}{P_{q,t}}\Big)^{\frac{-\epsilon}{\alpha}} Q_t^{\frac{-\epsilon}{\alpha}} = E_t(i)\Big[A\Big(\frac{P_{e,t}}{W_t}\frac{\alpha_\ell}{\alpha_e}\Big)^{\alpha_\ell}\Big(\frac{P_{e,t}}{r_t^k P_{k,t}}\frac{\alpha_k}{\alpha_e}\Big)^{\alpha_k}\Big]^{\frac{1}{\alpha}}.$$

By integrating out,

$$\Big(\int_{[0,1]} (\frac{P_{q,t}(i)}{P_{q,t}})^{\frac{-\epsilon}{\alpha}} di\Big)^{\alpha} Q_t = E_t^{\alpha} A\Big(\frac{P_{e,t}}{W_t}\frac{\alpha_\ell}{\alpha_e}\Big)^{\alpha_\ell}\Big(\frac{P_{e,t}}{r_t^k P_{k,t}}\frac{\alpha_k}{\alpha_e}\Big)^{\alpha_k}.$$

Recall that

$$\frac{W_t L_t(i)}{\alpha_\ell} = \frac{r_t^k P_{k,t} K_t(i)}{\alpha_k} = \frac{P_{e,t} E_t(i)}{\alpha_e}.$$

By taking integral, we get

$$\frac{W_t L_t}{\alpha_\ell} = \frac{r_t^k P_{k,t} K_t}{\alpha_k} = \frac{P_{e,t} E_t}{\alpha_e}.$$

Combining with (A.3), we have the result. □

Lemma A.2. *Under the Calvo price setting, the following "Aggregate Price Relationship" holds:*

$$P_{q,t} = \Big(\theta P_{q,t-1}^{1-\epsilon} + (1-\theta)(P_{q,t}^{o})^{1-\epsilon}\Big)^{\frac{1}{1-\epsilon}}.$$

Proof. By definition we have:

$$
\begin{aligned}
P_{q,t}^{1-\epsilon} &= \int\limits_{[0,1]} P_{q,t}(i)^{1-\epsilon} di \\
&= \int\limits_{\text{Firms that cannot change price}} P_{q,t}(i)^{1-\epsilon} di + \int\limits_{\text{Firms setting price optimally}} P_{q,t}(i)^{1-\epsilon} di \\
&= \int\limits_{[0,1]} \theta P_{q,t-1}(i)^{1-\epsilon} di + \int\limits_{[0,1]} (1-\theta) P_{q,t}(i)^{1-\epsilon} di \\
&= \theta P_{q,t-1}^{1-\epsilon} + (1-\theta)(P_{q,t}^{o})^{1-\epsilon}.
\end{aligned}
$$

□

Define $v_t := \int\limits_{[0,1]} \big(\frac{P_{q,t}(i)}{P_{q,t}}\big)^{\frac{-\epsilon}{\alpha}} di$.

Lemma A.3. *Under the Calvo price setting,*

$$
v_t = \theta v_{t-1} \Pi_{q,t}^{\frac{\epsilon}{\alpha}} + (1-\theta)\Big(\frac{P_{q,t}^{o}}{P_{q,t}}\Big)^{\frac{-\epsilon}{\alpha}}
$$

Proof. As Lemma A.2 □

Equilibrium

At equilibrium: **(i)** Each economic agent solves its maximization problem; **(ii)** All markets clear, i.e., the following equations hold:

$$
\begin{aligned}
\text{Capital:} \quad K_t &= \int\limits_{[0,1]} K_t(i) di, \\
\text{Labor:} \quad L_t &= \int\limits_{[0,1]} L_t(i) di, \\
\text{Energy:} \quad E_t &= \int\limits_{[0,1]} E_t(i) di, \\
\text{Resource constraint:} \quad P_{c,t}C_t + P_{k,t}I_t + G_t &= P_{q,t}Q_t - P_{e,t}E_t.
\end{aligned}
$$

(iii) And the government budget constraint is fulfilled:

$$
(1 + i_{t-1})B_{t-1} + G_t = B_t + T_t,
$$

Steady state

$$
\begin{array}{rrcl}
\text{Static problem of Household:} & C & = & \Theta_x C_e^x C_q^{1-x} \\
 & \dfrac{P_c}{P_{q,t}} & = & S_e^x =: S_c \\
 & C_q & = & (1-x) P_c^r C \\
 & S_e C_e & = & x S_c C. \\
 & 1 & = & \beta(r^k + 1 - \delta) \\
\text{Budget constraint:} & P_c^r C + \delta S_k K & = & W^r N + r^k S_k K + \Pi^r, \\
 & \Pi^r & = & Q - S_e E - W^r L - r^k S_k K \\
\text{The FOC for the representative household is:} & W_r & = & S_c C L^{\phi} \\
\text{Production function:} & Q & = & A E^{\alpha_e} L^{\alpha_\ell} K^{\alpha_k} \\
\text{The FOC for firms are :} & \dfrac{S_e E}{\alpha_e} & = & \dfrac{W^r L}{\alpha_\ell} = \dfrac{r^k S_k K}{\alpha_k} \\
 & S_e E & = & \dfrac{\alpha_e(\epsilon - 1)}{\epsilon} Q.
\end{array}
$$

Where $W_r = \frac{W}{P_q}$. Without loss of generality, we assume that:

$$
\begin{aligned}
S_e &= \frac{P_e}{P_q} = 1 \\
S_k &= \frac{P_k}{P_q} = 1
\end{aligned}
$$

We have to find $(C, C_e, C_q, r^k, W, Q, E, L, K)$.

Solution: Remember that $S_c C + \delta S_k K + G_r = Q - S_e E$, $E = \dfrac{\alpha_e(\epsilon-1)}{\epsilon} Q$, and $\dfrac{E}{\alpha_E} = \dfrac{r^k K}{\alpha_k}$, so that

$$
\begin{aligned}
C &= Q - S_e E - \delta S_k K - G \\
&= Q\Big(1 - \omega - \frac{\alpha_e(\epsilon-1)}{\epsilon} - \frac{\alpha_e(\epsilon-1)}{\epsilon}\frac{\delta\alpha_k}{\alpha_e r^k}\Big)
\end{aligned}
$$

Therefore, one can compute $\frac{Q}{C}$. The system of equations becomes

$$\begin{aligned}
r^k &= \frac{1}{\beta} - 1 + \delta \\
C &= \Big(1 - \omega - \frac{\alpha_e(\epsilon - 1)}{\epsilon}(1 + \delta\frac{\alpha_k}{\alpha_e r^k})\Big)Q \\
C_q &= (1 - x)C \\
C_e &= xC \\
W^r &= CL^\phi \\
Q &= AE^{\alpha_e}L^{\alpha_\ell}K^{\alpha_k} \\
\frac{E}{\alpha_e} &= \frac{W^r L}{\alpha_\ell} \\
E &= \frac{\alpha_e(\epsilon - 1)}{\epsilon}Q
\end{aligned}$$

By combining $W_r = CL^\phi$ with $\frac{E}{\alpha_e} = \frac{W_r L}{\alpha_\ell}$, we can compute the following quantities

$$\begin{aligned}
L^{\phi+1} &= \frac{(\epsilon - 1)\alpha_\ell}{\epsilon}\frac{Q}{C} \\
Q^{1-\alpha_e-\alpha_k} &= AL^{\alpha_\ell}\Big(\frac{\epsilon - 1}{\epsilon}\alpha_\ell\Big)^{\alpha_e}\big(\frac{\epsilon - 1}{\epsilon}\frac{\alpha_k}{r^k}\big)^{\alpha_k} \\
E &= \frac{\alpha_e(\epsilon - 1)}{\epsilon}Q \\
W_r &= \frac{\alpha_\ell(\epsilon - 1)}{L\epsilon}Q.
\end{aligned}$$

Remark: Oil's Cost Share and Oil's Output elasticity

Let us define the oil's cost share as follows:

$$\text{Oil's cost share} := \frac{P_e E}{P_c Y}$$

where Y is the GDP. Remember than in our case $P_c Y = P_q Q - P_e E$. Then

$$\begin{aligned}
\text{Oil's cost share} =& \frac{P_e E}{P_q Q - P_e E} \\
=& \frac{\frac{PeE}{PqQ}}{1 - \frac{P_e E}{P_q Q}}
\end{aligned}$$

at the steady state one has the following relationship:

$$\frac{P_e E}{P_q Q} = \frac{\alpha_e}{\mathcal{M}_p}$$

where $\mathcal{M}_p$ is the price markup and α_e is the output elasticity. Then one has:

$$\begin{aligned}\text{Oil's cost share} =& \frac{\frac{\alpha_e}{\mathcal{M}_p}}{1-\frac{\alpha_e}{\mathcal{M}_p}}\\ =& \frac{\alpha_e}{\mathcal{M}_p-\alpha_e}\end{aligned}$$

A.2 Log-linear Model

The model is log-linearized using the following rules:

- All variables in non-capital letter stand for the log-deviation, e.g : $e_t = \log(E_t) - \log(E)$, where variables without subscript stand for the steady state value.
- Exceptions for r_t^k, I_t and mc_t. We denote the log-deviation $\hat{r}_t^k$, $\hat{I}_t$ and $\hat{mc}_t$, respectively.
- All prices are in real value, in other words all prices ($P_{e,t}$, $P_{q,t}$, $P_{c,t}$ & $P_{k,t}$) are deflated by the core CPI ($P_{q,t}$).

The complete list of equations is:

$$
\begin{aligned}
i_t &= (1-\beta(1-\delta))\mathbb{E}_t[\hat{r}^k_{t+1}] + \mathbb{E}_t[\pi_{k,t+1}] \\
c_t &= \mathbb{E}_t[c_{t+1}] - (i_t - \mathbb{E}_t[\pi_{c,t+1}]) \\
w_{r,t} &= c_t + \phi l_t + x s_{e,t} \\
i_t &= \phi_\pi \pi_{q,t} + \phi_y y_t + \varepsilon_{i,t} \\
l_t + w_{r,t} &= s_{e,t} + e_t \\
&= \hat{r}^k_t + s_{k,t} + k_t \\
(S^x_e C)(c_t + x s_{e,t}) + (S_k I)(s_{k,t} + \hat{I}_t) &\ + G_r g_{r,t} = Qq_t - (S_e E)(s_{e,t} + e_t) \\
\delta \hat{I}_t &= k_{t+1} - (1-\delta)k_t \\
q_t &= a_t + \alpha_l l_t + \alpha_e e_t + \alpha_k k_t \\
\pi_{q,t} - \beta \mathbb{E}_t[\pi_{q,t+1}] &= \left(\frac{(1-\beta\theta)(1-\theta)\alpha}{\theta(\alpha + (1-\alpha)\epsilon)}\right) \hat{mc}_{r,t} + \varepsilon_{p,t} \\
\hat{mc}_{r,t} &= \left[\frac{1-\alpha}{\alpha} q_t + F^r_t\right] \\
F_{r,t} &= -\frac{1}{\alpha}(a_t - \alpha_e s_{e,t} - \alpha_l w_{r,t} - \alpha_k (r^k_t + s_{k,t})) \\
(S^x_e Y)(y_t + x s_{e,t}) &= Qq_t - (S_e E)(s_{e,t} + e_t) \\
\pi_{c,t} &= \pi_{q,t} + x \Delta s_{e,t} \\
\pi_{k,t} &= \pi_{q,t} + \Delta s_{k,t} \\
g_{r,t} &= \rho_g g_{r,t-1} + \rho_{ag} e_{a,t} + e_{g,t} \\
s_{e,t} &= \rho_{se} s_{e,t-1} + e_{se,t} \\
s_{k,t} &= \rho_{sk} s_{k,t-1} + e_{sk,t} \\
a_t &= \rho_a a_{t-1} + e_{a,t} \\
\varepsilon_{i,t} &= \rho_i \varepsilon_{i,t-1} + e_{i,t} \\
\varepsilon_{p,t} &= \rho_p \varepsilon_{p,t-1} + e_{p,t} - \nu_p e_{p,t-1}
\end{aligned}
$$

- $\varepsilon_{i,t}$ stands for the exogenous part of the monetary policy.
- $\varepsilon_{p,t}$ stands for the price mark-up disturbance, which we assume follows an ARMA(1,1). The inclusion of the MA part is designed to capture the high-frequency fluctuations in inflation.

This system has 20 variables and 20 equations. 14 endogenous variables, namely (i_t, $\hat{r}^k_t$, $\pi_{k,t}$, $\pi_{c,t}$, $\pi_{q,t}$, $w_{r,t}$, $\hat{mc}_{r,t}$, c_t, l_t, y_t, e_t, k_t, $\hat{I}_t$, q_t) and 6 exogenous disturbances (a_t, $s_{e,t}$, $s_{k,t}$, $g_{r,t}$, $\varepsilon_{i,t}$, $\varepsilon_{p,t}$).

A.3 Bayesian Estimation Procedure

Data Treatment

This section details the Bayesian estimation procedure of the DSGE model developed in the body of the paper.

We use six key macro-variables for our estimations. All series are quarterly. A description of the original series' sources is presented in Table A.1 and the data is available upon request. The sample goes from 1984:Q1 to 2007:Q1.

TABLE A.1: Original Sources

Serie	Description	Source
GDPC09	Real Gross Domestic Product, Chained Dollars (2009), Seasonally Adjusted, Annual Rate	Table 1.1.6 Bureau of Economic Analysis
GDPDEF	Implicit Price Deflators for Gross Domestic Product (2009), Seasonally Adjusted	Table 1.1.9. Bureau of Economic Analysis
PFI	Private Fixed Investment by Type, Seasonally Adjusted,Annual Rate	Table 5.3.5. Bureau of Economic Analysis
CE16OV	Civilian Employment, 16 and over, Seasonally Adjusted, Thousands	LNS12000000 Bureau of Labor Statistics
CE16OV Index	CE160V (2009)=1	
LNS10	Population level, civilian noninstitutional population, 16 and over, Seasonally Adjusted, Thousands	LNS10000000 Bureau of Labor Statistics
LNS10 Index	LNS10 (2009)=1	
PRS85006023	Nonfarm Business, All Persons, Average weekly hours worked Duration (2009), Seasonally Adjusted	PRS85006023 Bureau of Labor Statistics
FEDFUND	Federal funds effective rate, percent: Per Year, Average of Daily figures	Board of Governors of the Federal Reserve System
$O_{commercial}$	Total Petroleum Consumed by the Commercial Sector, Thousand barrels per day	Table 3.7a. U.S Energy Information Administration
$O_{industrial}$	Total Petroleum Consumed by the Industrial Sector, Thousand barrels per day	Table 3.7b. U.S Energy Information Administration
$O_{electrical}$	Total Petroleum Consumed by the Electrical Power Sector, Thousand barrels per day	Table 3.7c. U.S Energy Information Administration
$O_{transport}$	Total Petroleum Consumed by the Transport Sector, Thousand barrels per day	Table 3.7c. U.S Energy Information Administration
PSG	Passenger to freight, TBTu	Transportation Energy Intensity Indicators US Department of Energy

Our observable variables include: (i) real GDP, (ii) real Private Fixed Investment, (iii) hours worked, (iv) inflation (through the GDP price deflator), (v) the Federal Funds Rate and (vi) total oil use in production. The model is stationary, so we remove the trend of the first two series, that are trend stationary. The rest of the series are stationary, we do not remove their trends, but we take out their respective mean for the estimation period. A detailed explanation is presented on Table A.2.

Table A.2: Observable Variables

Observed Variable	Transformation
invobs	$detrend\left(ln\left(\frac{\frac{PFI}{GDPDEF}}{LNSIndex}\right)*100\right)$
yobs	$detrend\left(ln\left(\frac{GDPC09}{LNSIndex}\right)*100\right)$
labobs	$ln\left(\frac{PRS85006023*CE16OVIndex}{LNSIndex}\right)*100 - mean\left(ln\left(\frac{PRS85006023*CE16OVIndex}{LNSIndex}\right)*100\right)$
infobs	$ln\left(\frac{GDPDEF}{GDPDEF(-1)}\right)*100 - mean\left(ln\left(\frac{GDPDEF}{GDPDEF(-1)}\right)*100\right)$
iobs	$\left(ln\left(1+\frac{FEDFUND}{400}\right) - mean\left(ln\left(1+\frac{FEDFUND}{400}\right)\right)\right)*100$
eobs	$ln\left(\frac{TotalSAOil}{LNSIndex}\right)*100 - mean\left(ln\left(\frac{TotalSAOil}{LNSIndex}\right)*100\right)$

The total oil consumption of the production sector *TotalSAOil*, is constructed as follows:

$$TotalOil = O_{industrial} + O_{electrical} + O_{commercial} + (1 - PSG) * O_{transport},$$

where PSG is a measure of energy consumption in transport by households,[1] computed as the ratio of the energy consumption of all passengers and the total energy consumption in transport (Total Energy consumption in transport=energy consumption of all passengers + total energy consumption of All Freight).

Then, seasonality is removed with X12-ARIMA software from the Census Bureau, implemented in the open-source GRETL software, from where we obtain the series *TOTALSAOil*.

Finally, we have to identify our observable variables to our model's variables. Note that we have different prices in our model, among them: the domestic price, the CPI, which is equal to the GDP deflator by definition, and the price capital. Because we deflate the investment series by the GDP deflator (in the data treatment) and in our model the real

[1] As for oil, it is the source of some 95% of transport fuels globally, and without oil-based transport none of the other energy forms (such as electricity) and other primary energy sources (such as coal, gas, biomass, wind, solar, hydro, and so on) can be delivered. In this specific sense oil remains the most critical of all energy sources, in particular in transports.

series are deflated by the domestic price, we use the following observation equation for the investment:

$$invobs_t = \widehat{I}_t + s_{k,t} - xs_{e,t}$$

The other equations are:

$$\begin{aligned} ybos_t &= y_t \\ labobs_t &= l_t \\ eobs_t &= e_t \\ infobs_t &= \pi_{c,t} \\ iobs_t &= i_t \end{aligned}$$

Identification Analysis

In order to run an identification analysis, we need to specify starting values for all parameters. We first initialize our parameters as in Table A.3

TABLE A.3: Starting Values–First Identification

α_e	α_ℓ	α_k	ϕ	ϕ_π	ϕ_y	θ	ρ_j	σ_j
0.015	0.7	0.3	1.17	1.2	0.5	0.65	0.5	1

where $j \in \{a, se, sk, g, p, i\}$, so that ρ_j denotes all the autoregressive parameters in the model and σ_j, all the standard deviations.

The measure of identification strength developed by Iskrev (2010) and Andrle (2010) gives the following result

All parameters are identified in the model (rank of H). All parameters are identified by J moments (rank of J).

FIGURE A.1: Rank Condition

Figure (A.2) refers to the identification and sensitivity methodologies with respect to the first and second moments proposed by Iskrev (2010) and Andrle (2010). We remark that all parameters are identified, this result confirms the necessary and sufficient conditions (printed in Figure (A.1)) discussed by Iskrev (2010) for local identifiability. Nevertheless, we observe a lack of identifiability strength for the parameter α_e, here around 0.015.

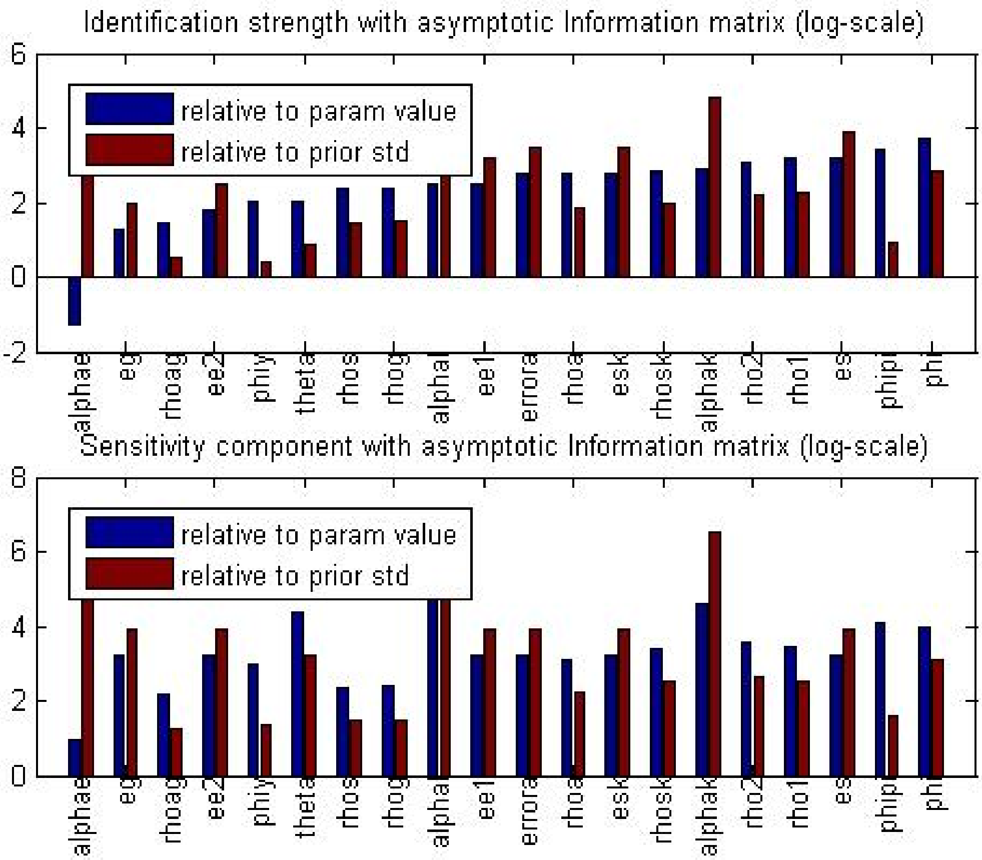

FIGURE A.2: Identification Strength

Then we test whether this identification strength issue could be fixed using different initial values for the elasticities.

Figures (A.3) to (A.8) summarize the identification strength explained *supra* for the set of initial values in Table A.4. Few observations of these graphs are worth making. Firstly, the higher α_e, the higher identification strength. Secondly, all parameters (except output elasticities) nearly keep the same ranking, in the sense that there is no shift greater than two positions. Thirdly, α_ℓ decreases in its identification strength, whereas α_k keeps the same ranking. Fourthly, one can note that in this experimentation, parameter θ, as opposed to the initial calibration, looses nearly all its identification strength. This explains why we estimate and compare the model with and without estimating θ.

TABLE A.4: Set of Starting Values

Elasticity	(1)	(2)	(3)	(4)	(5)	(6)
α_e	0.1	0.2	0.3	0.4	0.5	0.6
α_k	0.3	0.3	0.2	0.2	0.2	0.1
α_ℓ	0.7	0.6	0.6	0.5	0.4	0.4

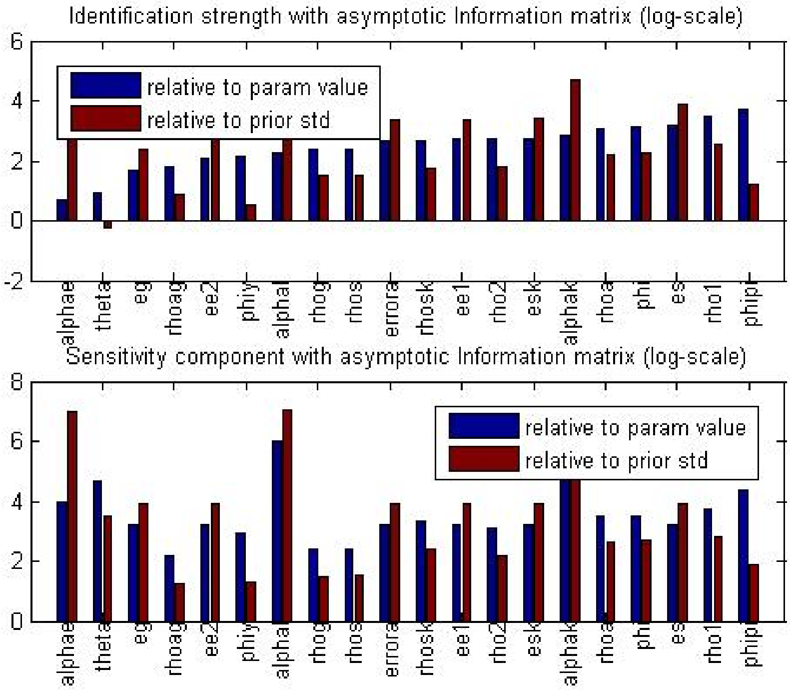

FIGURE A.3: Identification Strength for (1)

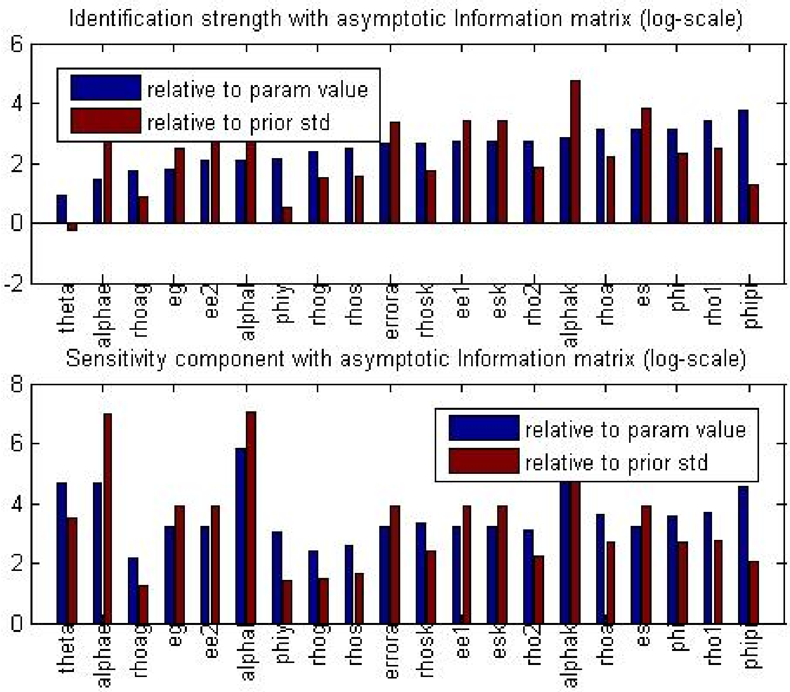

FIGURE A.4: Identification Strength for (2)

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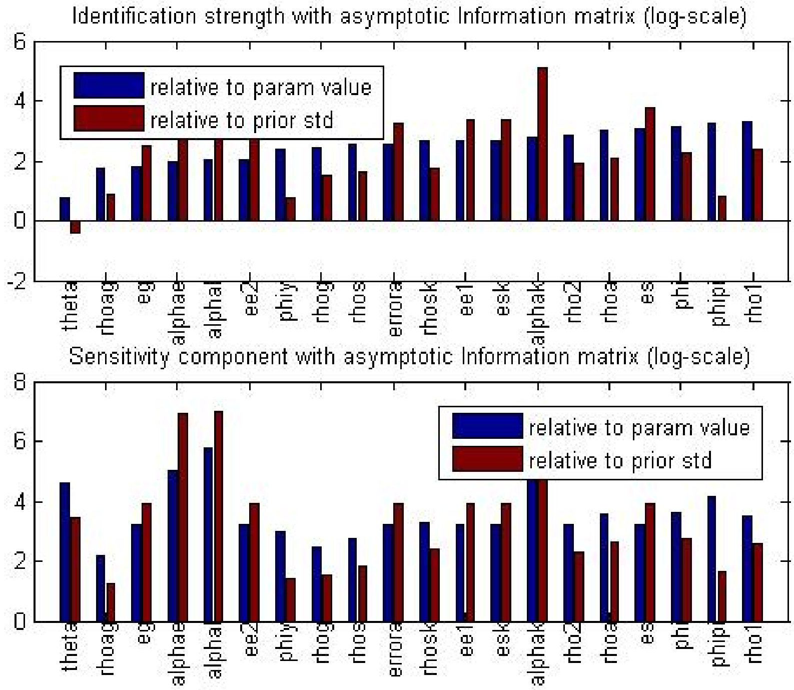

Figure A.5: Identification Strength for (3)

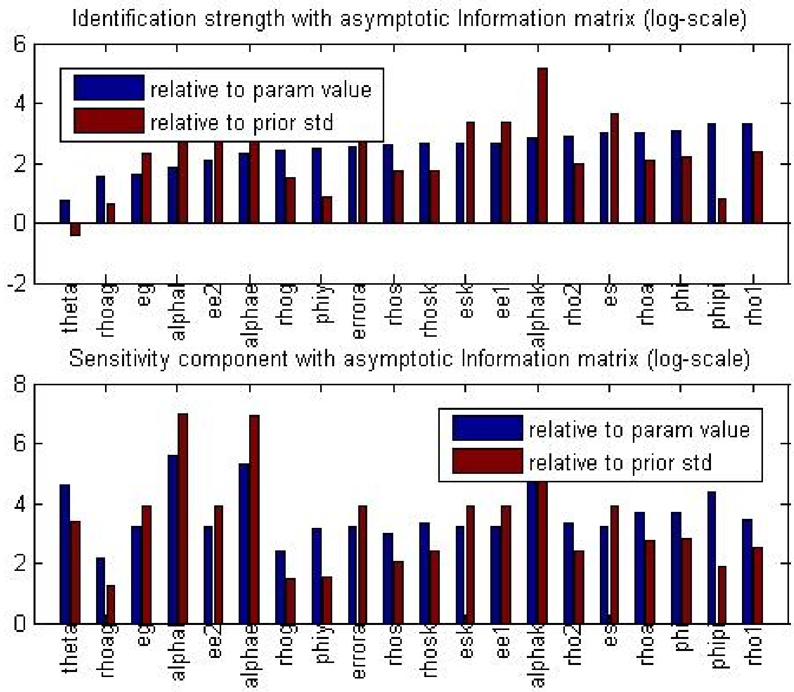

Figure A.6: Identification Strength for (4)

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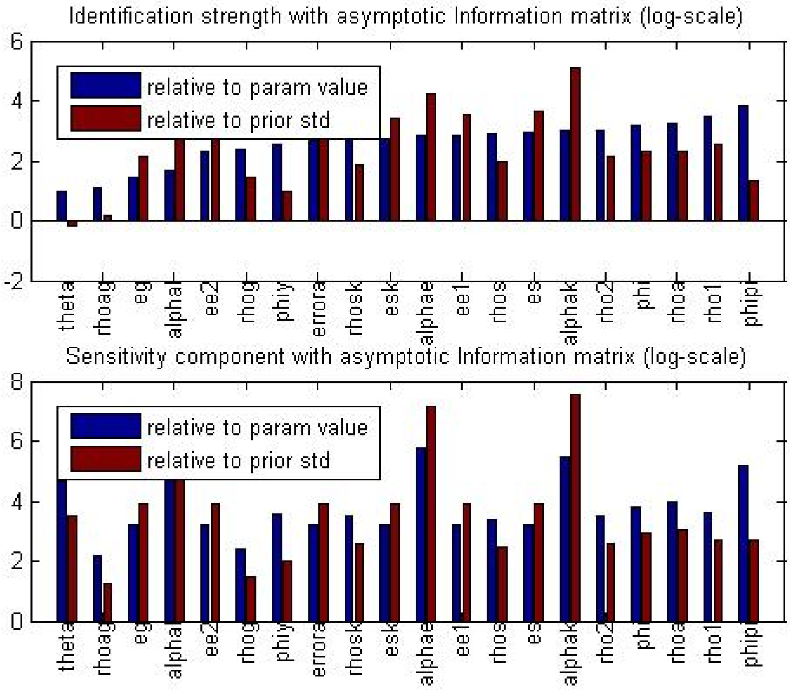


FIGURE A.7: Identification Strength for (5)

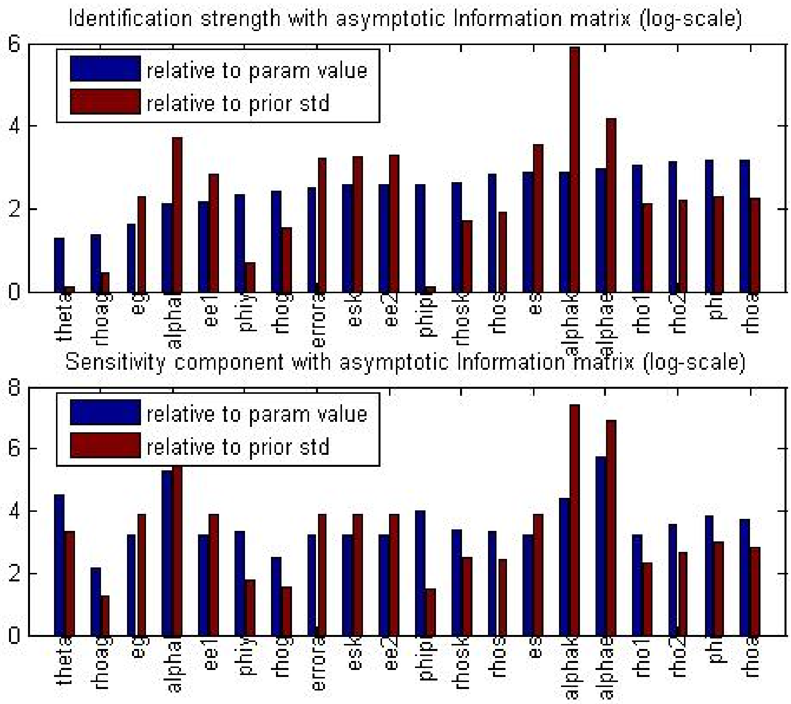


FIGURE A.8: Identification Strength for (6)

A.4 Estimation Results

There are 26 parameters, including parameters that characterize the exogenous shocks. As explained in Chapter 2, we fix 5 of them according to the literature. The calibration of these parameters are resumed in Table A.5. Those parameters are calibrated due to their well-known lack of identification in macro-data. Note that for estimation proposes we add an ad-hoc shock for the New-Keynesian Phillips Curve $\varepsilon_{p,t}$[2] that can be interpreted as being a markup shock.

Table A.5: Calibrated Parameters

β	δ	ω	x	ϵ
0.99	0.025	0.18	0.023	8

The parameters' priors of other variables remain unchanged except for the priors of α_k and α_ℓ that change along with α_e, as shown in Table A.4.

Table A.6: Estimation Results Summary

α_e prior value		Log marg. density		$\hat{\alpha}_e$		Sum of α_i	
θ estim.	θ calib.	θ estim.	θ calib.	θ estim.	θ calib.	θ estim.	θ calib.
0.015	0.015	-567.16	-570.93	0.1178	0.0117	1.3648	1.1077
0.3	0.5	-567.65	-589.99	0.085	0.1177	1.3622	1.0952
0.5	0.2	-579.18	-591.80	0.1138	0.0533	1.2002	1.0913
0.6	0.1	-586.98	-592.99	0.1254	0.0356	1.1264	1.1188
0.1	0.6	-592.84	-593.28	0.082	0.1304	1.1168	1.0966
0.4	0.3	-596.08	-596.51	0.1090	0.0625	1.0226	1.1023
0.2	0.4	-596.92	-600.66	0.0839	0.1055	1.1322	1.0915

Table A.6 is ranking (ascending) with respect to the log-marginal density values. Several observations can emerge from this table. First, the first estimate of oil's output elasticity in the case θ calibrated, suggests an estimated $\hat{\alpha}_e$ similar to its first prior value, i.e $\alpha_e = 0.015$. Then the identification strength of this parameter advocates a weak identification. This intuition is confirmed using Figure A.9: the prior and the posterior distribution match, therefore the weak identification of α_e is confirmed since this parameter is only explained by its prior distribution. Thus, for the case θ calibrated, the best log-marginal density is obtained is when we assume that the prior mean for α_e is 0.5.

[2]Where $\varepsilon_{p,t}$ is a $ARMA(1,1)$ process of the form $\varepsilon_{p,t} = \rho_p \varepsilon_{p,t-1} + e_{p,t} - \nu_p e_{p,t-1}$, where $e_{p,t} \sim \mathcal{N}(0, \sigma_p^2)$.

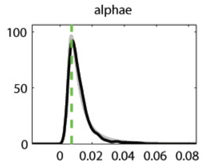


FIGURE A.9: Prior and Posterior Distribution of α_e

Posterior (solid black kine) and prior (solid grey line) distribution of α_e. The dashed green line stands for the posterior empirical mode

Second, the first (best) three estimations in the case θ estimated give us the sum of elasticities greater than the steady state markup ($\varepsilon/(\varepsilon-1) \approx 1.14$). Then there is a problematic economic interpretation due to the fact that one can show that if $\sum_{i\in\{e,l,k\}} \alpha_i > \varepsilon/(\varepsilon-1)$, the steady state value of firm's profit is negative. This is not surprising, since the model does not restrict the production function to have a constant return to scale technology together with the fact that the estimation procedure can hit the upper bound of the prior distribution. So in this case, one might find results without economic sense. In order to avoid this situation, we propose a narrower restriction on the upper bound of prior distribution on output elasticities, define shortly.

Restricted Estimation

Table A.7 refers to the upper bound restriction limits for the first three estimations in Table A.6, for the case θ estimated.

TABLE A.7: Prior's Upper Bound Restriction on Output Elasticities Parameters

Elasticity	0.015	0.3	0.5
α_e	0.4	0.4	0.3
α_k	0.3	0.35	0.3
α_ℓ	0.7	0.75	0.7

As shown in Table A.8, once we restrict the model, the log-marginal density of this estimations drops to a lower level. Then for the case θ estimated, the best log-data density is obtained when we assume that the prior mean for α_e is 0.6, which corresponds to the estimation in forth column of the Table A.6.

Results

As for the results obtained regarding the estimates of the stochastic processes resumed in Table A.9, one can extract important observations. Concerning standard deviation

Table A.8: Estimation Results for Restricted Parameters

α_e prior value	Log marg. density	$\hat{\alpha}_e$	Sum of α_i
0.015	-591.24	0.0798	1.0666
0.3	-594.37	0.0727	1.0681
0.5	-620.28	0.1458	1.1341

estimates, most of the variance is driven by the demand shock (σ_g) and real price of oil (σ_{se}) in both cases.[3] The high standard deviation for the price of oil can be interpreted as being the resulting of a financial asset trade in a volatile stock market. For the case θ calibrated, we find a high persistency on $AR(1)$ coefficients for government spending (0.93), price markup (0.96), technology (0.94) and the real price of oil (0.98), whereas for the other case, only the first two, together with the monetary policy (0.9308) have a high autoregressive parameter.

[3]Note that standard deviations describe in Table A.9 are in percentage, meaning that if $\sigma = 1$, then 1 stands for 1%

Table A.9: Prior and Posterior Distribution of Shock Parameters

Parameter		Prior distribution	Posterior distribution			
			Mode	Mean	10%	90%
θ estimated						
Autoregressive parameters						
Technology	ρ_a	Beta(0.5,0.2)	0.8619	0.8481	0.7960	0.8999
Real oil price	ρ_{se}	Beta(0.5,0.2)	0.5761	0.5611	0.4629	0.6669
Real capital price	ρ_{sk}	Beta(0.5,0.2)	0.7210	0.7080	0.6647	0.7524
Price markup1	ρ_p	Beta(0.5,0.2)	0.9418	0.9283	0.8955	0.9640
Price markup2	ν_p	Beta(0.5,0.2)	0.9796	0.9760	0.9610	0.9913
Government	ρ_g	Beta(0.5,0.2)	0.9058	0.8995	0.8712	0.9258
Tech. in Gov.	ρ_{ag}	Beta(0.5,0.2)	0.6904	0.6127	0.3549	0.9472
Monetary	ρ_i	Beta(0.5,0.2)	0.9399	0.9308	0.9035	0.9581
Standard deviations						
Technology	σ_a	IGamma(1,2)	0.4361	0.4435	0.3901	0.4942
Real oil price	σ_{se}	IGamma(1,2)	2.0000	1.9373	1.8652	2.000
Real capital price	σ_{sk}	IGamma(1,2)	0.7740	0.7675	0.6379	0.8781
Price markup	σ_p	IGamma(1,2)	0.1814	0.1854	0.1615	0.2094
Government	σ_g	IGamma(1,2)	2.0000	1.7921	1.5508	1.9998
Monetary	σ_i	IGamma(1,2)	0.5410	0.4566	0.3859	0.5205
θ calibrated						
Autoregressive parameters						
Technology	ρ_a	Beta(0.5,0.2)	0.9605	0.9401	0.9033	0.9774
Real oil price	ρ_{se}	Beta(0.5,0.2)	0.9934	0.9872	0.9754	0.9977
Real capital price	ρ_{sk}	Beta(0.5,0.2)	0.8940	0.8924	0.8483	0.9314
Price markup1	ρ_p	Beta(0.5,0.2)	0.9839	0.9621	0.9299	0.9971
Price markup2	ν_p	Beta(0.5,0.2)	0.1652	0.1711	0.0593	0.2758
Government	ρ_g	Beta(0.5,0.2)	0.9373	0.9312	0.9061	0.9560
Tech. in Gov.	ρ_{ag}	Beta(0.5,0.2)	0.7129	0.6589	0.3808	0.9541
Monetary	ρ_i	Beta(0.5,0.2)	0.1914	0.2104	0.1249	0.2856
Standard deviations						
Technology	σ_a	IGamma(1,2)	0.4538	0.4542	0.3981	0.5078
Real oil price	σ_{se}	IGamma(1,2)	2.0000	1.9475	1.8842	2.000
Real capital price	σ_{sk}	IGamma(1,2)	0.5459	0.5750	0.4722	0.6714
Price markup	σ_p	IGamma(1,2)	0.4235	0.4645	0.2868	0.6602
Government	σ_g	IGamma(1,2)	2.0000	1.8359	1.6425	2.000
Monetary	σ_i	IGamma(1,2)	0.4778	0.4769	0.4062	0.54555

Appendix B

Appendix Chapter 3

Appendices are fourfold: (i) to compute the derivation of the stochastic models; (ii) to introduce extensively the estimation technique named the simulated maximum likelihood (hereafter SMLE) illustrated by an example; (iii) to emphasize numerical problems of the inference of the short term Phillips curve and; (iv) to display additional results on the backtesting strategy with data from Mohun and Veneziani (2006).

B.1 Models Derivation

This appendix will introduce only the stochastic models. The deterministic counterparts will be deduce from the stochastic. The Goodwin-predator-prey model(see Goodwin (1967)) will be first presented, the extension of Van der Ploeg (1985) will follow.

B.1.1 The Stochastic Predator-Prey Model

Goodwin endows the productive sector with a Leontief production function

$$Y_t = \min\left(a_t L_t, \frac{K_t}{\nu}\right),$$

where Y_t is the real output, L_t is the employed population, a_t, the labor productivity, K_t is the stock of capital, and ν is the constant capital-to-output ratio. By assuming full capacity utilization, the following equality holds:

$$Y_t = a_t L_t = \frac{K_t}{\nu}.$$

The labor productivity is assumed to grow according to a stochastic process

$$\frac{da_t}{a_t} = \alpha dt - \sigma_1(.)dB_t^1,$$

where $\sigma_1(.)$ is the diffusion function (the arguments can be all the state variables) and B_t^1 is a Brownian motion. For the sake of clarity, the remaining of the model derivation is divided into two subsections, the first for the wage share and the second for the employment rate.

B.1.1.1 The Wage Share

Real wages growth is assumed to grow according to the stochastic differential equation (hereafter SDEs)

$$\frac{dW_t}{W_t} = \Phi(\lambda_t)dt + \sigma_2(.)dB_t^2,$$

where $\Phi(.)$ is a smooth function, $\sigma_2(.)$ the diffusion process of B_t^2 a Brownian motion assumed to be orthogonal to B_t^1. The wage share, ω, is defined as

$$\omega_t := \frac{W_t L_t}{Y_t} = \frac{W_t}{a_t}.$$

Using the multidimensional version of the îto lemma for the function $f(x, y) = x/y$, one has

$$\frac{d\omega_t}{\omega_t} = \left(\Phi(\lambda_t) - \alpha + \sigma_1^2(.)\right) dt + \sigma_1(.)dB_t^1 + \sigma_2(.)dB_t^2. \tag{B.1}$$

B.1.1.2 The Employment Rate

The total labor force, N, is assumed to grow exogenously so that

$$\frac{dN_t}{N_t} = \beta dt.$$

The employment rate, λ, is defined as

$$\lambda_t := \frac{L_t}{N_t}.$$

The capital accumulates according to

$$\frac{dK_t}{K_t} = \left(\frac{I_t}{K_t} - \delta\right) dt,$$

where I_t is the investment and δ is the depreciation rate of capital. Profits, Π, is defined as

$$\Pi_t := Y_t - W_t L_t$$

Thus, the profit rate (profit-to-output ratio) is defined by

$$\pi_t := \frac{\Pi_t}{Y_t} = 1 - \omega_t.$$

If one assumes that profits equal investment ($I_t = \Pi_t$),

$$\frac{dK_t}{K_t} = \left(\frac{(1-\omega_t)}{\nu} - \delta \right) dt.$$

By re-writting λ_t, one has

$$\lambda_t = \frac{L_t}{N_t} = \frac{K_t}{\nu a_t N_t}.$$

Using îto lemma for the function $f(x, y, z) = x/(\nu y z)$[1] one has

$$\frac{d\lambda_t}{\lambda_t} = \left[\frac{(1-\omega_t)}{\nu} - (\alpha + \beta + \delta) - \sigma_1^2(.) \right] dt + \sigma_1(.) dB_t^1. \tag{B.2}$$

Equations (B.1) and (B.2) make the two-dimensional stochastic prey-predator model. One can note that if $\sigma_1(.) = \sigma_2(.) = 0$, the system becomes the deterministic prey-predator model.

B.1.2 The Stochastic van der Ploeg (1985)'s Extension

Suppose that the productive sector is endowed with a CES production technology so that

$$Y = C \left[\pi K^{-\eta} + (1-\pi)(\lambda^L L)^{-\eta} \right]^{-\frac{1}{\eta}} \tag{B.3}$$

where λ^L, the labor productivity, follows the stochastic process

$$\frac{\dot{\lambda}^L}{\lambda^L} = \alpha dt - \sigma_1 dB_t^1. \tag{B.4}$$

Assuming that the real wage, W, is set at its marginal rate,

$$\frac{\partial Y}{\partial L} = W. \tag{B.5}$$

[1]Computation details will be a special case of the van der Ploeg model in the next Section.

For simplicity, one can consider that $L^e := \lambda^L L$, then

$$\frac{\partial Y}{\partial L^e} = \frac{\partial Y}{\partial L}\frac{1}{\lambda^L}. \tag{B.6}$$

Using equations (B.5) and (B.3),

$$\frac{\partial Y}{\partial L^e} = \frac{(1-\pi)}{C^\eta}\left(\frac{Y}{L^e}\right)^{1+\eta}.$$

Equalizing equations (B.5) and (B.6) through (B.7) :

$$\begin{aligned} \left(\frac{\omega}{1-\pi}\right)^{\frac{1}{\eta}} C &= \frac{Y}{L^e} \\ \Leftrightarrow \left(\frac{\omega}{1-\pi}\right)^{\frac{1}{\eta}} C\lambda^L &= \frac{Y}{L} \end{aligned} \tag{B.7}$$

B.1.2.1 Wage share

As previously, ω is the real wage share

$$\omega := \frac{WL}{Y}.$$

With $a := Y/L$, this equality holds $\omega = W/a$. Until now, we do not know the dynamic of ω. The idea is to assume a SDEs for ω, so that

$$d\omega_t = \omega_t(f dt + g_1 dB_t^1 + g_2 dB_t^2). \tag{B.8}$$

One has to identify the f, g^1 and g^2 functions. As previously, the real wage growth evolves according to

$$dW = W(\phi(\lambda_t)dt + \sigma_2 dB_t^2). \tag{B.9}$$

Or, from equation (B.7), one has

$$a_t = \left(\frac{\omega_t}{1-\pi}\right)^{\frac{1}{\eta}} C\lambda_t^L,$$

where ω is defined by equation (B.8) and λ^L by equation (B.4). Using the îto formula for the function $a = f(\omega, \lambda^L)$ one has

$$\frac{da_t}{a_t} = \frac{1}{\eta}\frac{d\omega_t}{\omega_t} + \frac{d\lambda_t^L}{\lambda_t^L} + \left(\frac{1-\eta}{\eta^2}\frac{(g_1^2+g_2^2)}{2} - \frac{1}{\eta}\sigma_1 g_1\right)dt. \tag{B.10}$$

This result is obtain by assuming that the two Brownian motions, B_t^1 and, B_t^2, are independent. Applying once again the îto formula for ω with equations (B.10) and (B.9)

$$d\omega_t = \omega_t \left(\frac{dW_t}{W_t} - \frac{da_t}{a_t} - \frac{d < a, W >_t}{a_t W_t} + \frac{d < a, a >_t}{a_t^2} \right). \tag{B.11}$$

Therefore,

$$\begin{aligned}\frac{d\omega_t}{\omega_t} &= \underbrace{\phi(\lambda_t) - \frac{f}{\eta} - \alpha - \frac{1}{2}\left(\frac{1-\eta}{\eta^2}(g_1^2 + g_2^2) - \frac{1}{\eta}\sigma_1 g_1\right) - \frac{\sigma_2 g_2}{\eta} + \left(\frac{g_1}{\eta} - \sigma_1\right)^2 + \left(\frac{g_2}{\eta}\right)^2}_{=f} dt \\ &+ \underbrace{\left(\sigma_1 - \frac{g_1}{\eta}\right)}_{=g_1} dB_t^1 + \underbrace{\left(\sigma_2 - \frac{g_2}{\eta}\right)}_{=g_2} dB_t^2.\end{aligned}$$

Thus,

$$\begin{aligned} g_1 &= \left(\frac{\eta}{\eta+1}\right)\sigma_1 \\ g_2 &= \left(\frac{\eta}{\eta+1}\right)\sigma_2 \\ f &= \left(\frac{\eta}{\eta+1}\right)\left\{\phi(\lambda_t) - \alpha - \frac{1}{2}\left(\frac{1-\eta}{(1+\eta)^2}(\sigma_1^2+\sigma_2^2) - \frac{\sigma_1^2}{\eta+1}\right)\right. \\ &- \left.\frac{\sigma_2^2}{\eta+1} + \left(\frac{\sigma_1\eta}{1+\eta}\right)^2 + \left(\frac{\sigma_2}{1+\eta}\right)^2\right\}. \end{aligned}$$

Finally, the wage share dynamic is

$$\begin{aligned}\frac{d\omega_t}{\omega_t} &= \left(\frac{\eta}{\eta+1}\right)\left\{\phi(\lambda_t) - \alpha - \frac{1}{2}\left(\frac{1-\eta}{(1+\eta)^2}(\sigma_1^2+\sigma_2^2) - \frac{\sigma_1^2}{\eta+1}\right)\frac{\sigma_2^2}{\eta+1} + \left(\frac{\sigma_1\eta}{1+\eta}\right)^2 + \left(\frac{\sigma_2}{1+\eta}\right)^2\right\} dt \\ &+ \left(\frac{\eta}{\eta+1}\right)\sigma_1 dB_t^1 + \left(\frac{\eta}{\eta+1}\right)\sigma_2 dB_t^2 \end{aligned} \tag{B.12}$$

One can note that if $\eta \to +\infty$, capital and labor do not substitute, i.e. we retrieve the Leontief case previously defined,

$$\frac{d\omega_t}{\omega_t} = \left(\phi(\lambda_t) - \alpha + \sigma_1^2\right) dt + \sigma_1 dB_t^1 + \sigma_2 dB_t^2$$

B.1.2.2 The Employment rate

As previously mentioned, the labor force grows so that

$$\frac{dN_t}{N_t} = \beta dt$$

The employment rate is defined by $\lambda_t := L_t/N_t$, the following dynamic holds

$$\frac{d\lambda_t}{\lambda_t} = \frac{dL_t}{L_t} - \frac{dN_t}{N_t}.$$

This is a consequence of equation (B.11) and the fact that N, the total labor force, is deterministic. The capital accumulates so that

$$\frac{dK_t}{K_t} := \left(\frac{(1-\omega_t)}{\nu_t} - \delta\right) dt \tag{B.13}$$

Using equations (B.3) and (B.5), the capital-to-output ratio is such that

$$\nu_t = \left(\frac{1-\omega_t}{\pi}\right)^{-\frac{1}{\eta}} \frac{1}{C}.$$

Thus, the accumulation of capital can be written

$$\frac{dK_t}{K_t} := \left(C\pi^{-1/\eta}(1-\omega_t)^{1+1/\eta} - \delta\right) dt. \tag{B.14}$$

Using the îto lemma for the function $f(\omega_t) = \nu_t$, the capital-to-output ratio evolves so that

$$\frac{d\nu_t}{\nu_t} = \frac{1}{\eta}\frac{\omega_t}{1-\omega_t}\frac{d\omega_t}{\omega_t} + \left(\frac{\omega_t}{1-\omega_t}\right)^2 \left(\frac{1}{1+\eta}\right)\left(\frac{\sigma_1^2+\sigma_2^2}{2}\right) dt. \tag{B.15}$$

Again, by using the îto formula for $L = K/(\nu a)$–the function will be $L = f(K, \nu, a) = K/(\nu a)$. One has

$$\begin{aligned} \frac{dL}{L} &= \frac{dK}{K} - \frac{d\nu}{\nu} - \frac{da}{a} - \frac{d < K, a >_t}{Ka} - \frac{d < K, \nu >_t}{K\nu} + \frac{d < \nu, a >_t}{\nu a} \\ &+ \frac{d < a, a >_t}{aa} + \frac{d < \nu, \nu >_t}{\nu\nu} + \frac{d < K, K >_t}{KK}, \end{aligned}$$

thus,

$$\begin{aligned}
\frac{dL_t}{L_t} &= \left(C\pi^{-1/\eta}(1-\omega_t)^{1+1/\eta}-\delta\right)dt \\
&- \frac{1}{\eta}\frac{\omega_t}{1-\omega_t}\frac{d\omega_t}{\omega_t} - \left(\frac{\omega_t}{1-\omega_t}\right)^2\left(\frac{1}{1+\eta}\right)\left(\frac{\sigma_1^2+\sigma_2^2}{2}\right)dt \\
&- \frac{1}{\eta}\frac{d\omega_t}{\omega_t} - \alpha + \sigma_1 dB_t^1 - \left(\frac{1-\eta}{(1+\eta)^2}\frac{(\sigma_1^2+\sigma_2^2)}{2} - \frac{\sigma_1^2}{\eta+1}\right)dt \\
&- 0 - 0 + \left(\frac{\omega_t}{1-\omega_t}\right)\left\{-\eta\left(\frac{\sigma_1}{1+\eta}\right)^2 + \left(\frac{\sigma_2}{1+\eta}\right)^2\right\}dt \\
&+ \left(\frac{\eta}{\eta+1}\right)^2\sigma_1^2 dt + \left(\frac{\sigma_2}{1+\eta}\right)^2 dt \\
&+ \left[\left(\frac{\omega_t}{1-\omega_t}\frac{1}{1+\eta}\right)\right]^2(\sigma_1^2+\sigma_2^2)dt + 0.
\end{aligned}$$

The employment rate's SDEs is

$$\begin{aligned}
\frac{d\lambda_t}{\lambda_t} &= \left(C\pi^{-1/\eta}(1-\omega_t)^{1+1/\eta}-(\delta+\beta+\alpha)\right)dt \\
&- \left(\frac{\omega_t}{1-\omega_t}\right)^2\left(\frac{1}{1+\eta}\right)\left(\frac{\sigma_1^2+\sigma_2^2}{2}\right)dt \\
&- \left(\frac{1-\eta}{(1+\eta)^2}\frac{(\sigma_1^2+\sigma_2^2)}{2} - \frac{\sigma_1^2}{\eta+1}\right)dt \\
&+ \left(\frac{\omega_t}{1-\omega_t}\right)\left\{-\eta\left(\frac{\sigma_1}{1+\eta}\right)^2 + \left(\frac{\sigma_2}{1+\eta}\right)^2\right\}dt \\
&+ \left(\frac{\eta}{\eta+1}\right)^2\sigma_1^2 dt + \left(\frac{\sigma_2}{1+\eta}\right)^2 dt \\
&+ \left[\left(\frac{\omega_t}{1-\omega_t}\frac{1}{1+\eta}\right)\right]^2(\sigma_1^2+\sigma_2^2)dt - \frac{1}{\eta}\left(\frac{d\omega_t}{\omega_t(1-\omega_t)}\right) + \sigma_1 dB_t^1
\end{aligned} \tag{B.16}$$

The stochastic van der Ploeg model is entirely defined with equations (B.16) and (B.12). It is worth mentioning that if $\eta \to +\infty$ and $C := 1/\nu$ where ν is the constant capital-output ratio, one has

$$\frac{d\lambda_t}{\lambda_t} = \left(\frac{1-\omega_t}{\nu} - (\alpha+\beta+\delta) + \sigma_1^2\right)dt + \sigma_1 dB_t^1$$

Furthermore, if the model is deterministic ($\sigma_1 = \sigma_2 = 0$), we retrieve a model close to the one of Grasselli and Maheshwari (2016), i.e. :

$$\frac{d\lambda_t}{\lambda_t} = \left(C\pi^{-1/\eta}(1-\omega_t)^{1+1/\eta} - (\delta+\beta+\gamma) - \frac{1}{\eta}\left(\frac{d\omega_t}{\omega_t(1-\omega_t)}\right)\right)dt$$

B.2 The SMLE Method

The appendix explains in detail the SMLE method.

B.2.1 Notations

In what follows, one can consider a multivariate stochastic differential equations (hereafter : SDEs), on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ of the form

$$dX_t = f(X_t)dt + g(X_t)dB_t \tag{B.17}$$

Where

- $X_t \in \mathbb{R}^n$
- B_t is a d-dimensional Brownian motion.
- $f : \mathbb{R}^n \to \mathbb{R}^n$
- $g : \mathbb{R}^n \to \mathbb{R}^{n \times d}$, where $\forall x$, $g^{-1}(x)g(x)$ is positive definite.

B.2.2 Numerical Simulation of the Solution

Analytic solution of economic models are unlikely available. To approximate the numerical solution of model (B.17), I use the generalization of the Euler explicit method for ordinary differential equations to stochastic differential equations, namely the Euler-Maruyama scheme. For SDEs, several manners of approximating the solution exist. For instance the Jimenez et al. (1999) scheme. Despite the fact that Euler-type scheme is known to be less efficient than the others, this scheme is computationally stable at any case.

If one considers the model (B.17), with the initial condition $X_0 = x_o$, suppose that one wishes to solve the SDEs on some interval of time $[0, T]$. Then the Euler–Maruyama approximation to the true solution X is the Markov chain Y defined by

- Consider a partition of the interval $[0, T]$ into N equal subintervals of width $\Delta t > 0$:

$$0 = \tau_0 < \tau_1 < \cdots < \tau_N = T \text{ and } \Delta t = T/N.$$

- Set the initial condition $Y0 = x0$.

- Recursively one can define Y_n, for $1 \leq n \leq N$, by

$$Y_{n+1} = Y_n + f(Y_n)\,\Delta t + g(Y_n)\,\Delta W_n, \tag{B.18}$$

where

$$\Delta W_n = W_{\tau_{n+1}} - W_{\tau_n}.$$

The random variables ΔWn are independent and identically distributed normal random variables with expected value zero and variance Δt.

B.2.3 The Estimation : Simulated Maximum Likelihood Estimation

B.2.3.1 Overview

The methodology is borrowed from Durham and Gallant (2002), and is extended to the multivariate analysis.

If one writes the joint likelihood function as being $p(x_1, \ldots, x_T)$, where the observations are $x_i \in R^n, \forall i \in \{1, \ldots, T\}$, one can rewrite the likelihood function as being:[2]

$$p(x_1, \ldots, x_T) = p(x_1) \prod_{i=2}^{T} p(x_i, i; x_{i-1}, i-1)$$

The objective of the following is to give methodology to compute $p(x_t, t; x_s, s)$, in other words the transition probability of the process x from time s to time t . The first order approximation $p^{(1)}(x_t, t; x_s, s)$ defined by (B.18) will be accurate if the interval $[s, t]$ is short enough. Otherwise one may partition the interval such that the first-order approximation is sufficiently accurate on each subinterval ($s = \tau_0 < \ldots < \tau_M = t$). The random variables x_{τ_i} are unobserved, and must be integrated out. Because the process is Markovian, one obtains

$$\begin{aligned} p(x_t, t; x_s, s) &\approx p^{(M)}(x_t, t; x_s, s) \\ &:= \int \prod_{m=0}^{M-1} p^{(1)}(u_{m+1}, \tau_{m+1}; u_m, \tau_m) d\lambda(u_1, \ldots, u_{M-1}) \end{aligned}$$

where λ is here the Lebesgue measure, and the conventions $u_0 = x_s$, and $u_M = x_t$ are used. Monte Carlo integration is generally the only feasible way to evaluate the integral. For $s < t$ suppose that $x_t|x_s$ has a transition density $p(x_t, t; x_s, s)$ and let

$$p^{(1)}(x_t, t; x_s, s) = \phi(x_t; x_s + f(x_s)(t-s), g(x_s)\sqrt{(t-s)})$$

[2]The first element of the likelihood, $p(x_1)$, is unknown and will be neglected in the computation of the likelihood.

where $\phi(x, f, g)$ is the Gaussian density, be its first-order approximation. One can prove that, under mild assumptions[3] reported on Durham and Gallant (2002),

$$\lim_{M\to+\infty} p^{(M)}p(., t; x_s, s, \theta) = p(., t; x_s, s, \theta) \text{ , in } L^1(\lambda) \tag{B.19}$$

Pedersen (1995b) and Pedersen (1995a) showed that the convergence presented above is reach for the linear case. To the best of our knowledge, no proof has be made with non-linear functions nor counterexample as been found. In the paper, each time a nonlinear form is used, it has been tested using some data generating process (hereafter DGP).

B.2.3.2 How to Compute the Integral?

Let $\{u_k = (u_{k,1}, \ldots, u_{k,M-1}), k = 1, \ldots, K\}$ be independent draws from q–an importance sampler. One can define

$$p^{(M,K)}(x_t, t; x_s, s, \theta) = \frac{1}{K}\sum_{k=1}^{K} \frac{\prod_{m=1}^{M} p^{(1)}(u_{k,m}, \tau_m; u_{k,m-1}, \tau_{m-1}, \theta)}{q(u_{k,1}, \ldots, u_{k,M-1})} \tag{B.20}$$

where $u_{k,0} = x_s$ and $u_{k,m} = x_t$ for all k. Under some mild assumptions and the strong law of large numbers, on has

$$\lim_{K\to+\infty} |p^{(M,K)}(x_t, t; x_s, s, \theta) - p^{(M)}(x_t, t; x_s, s, \theta)| = 0 \text{ a.s.}$$

Durham and Gallant (2002) made the remark that when M is increasing, for a fix K, the bias will be reduced but the variance will increase. One may increase sufficiently K in order to reduce that variance but it is costly since the variance decreases at the speed $1/\sqrt{K}$.

B.2.3.3 Which Importance Sampler to choose?

The importance sampler that will be used is the one which draws u_{m+1} from a Gaussian density based on the first approximation conditional on u_m and x_t. That is, treating u_m and $u_M = x_t$ as fixed, one draws u_{m+1} from the density

$$\begin{aligned} p(u_{m+1}|u_m, u_M) &= p(u_{m+1}|u_m)p(u_M|u_{m+1})/p(u_M|u_m) \\ &= \phi(u_{m+1}; u_m/\tilde{\mu}_m\delta, \tilde{\sigma}_m^2\delta) \end{aligned}$$

[3] Including a nonexploding, unique weak solution of (3.5).

where $\delta = (t-s)/M$, and

$$\tilde{\mu}_m = \left(\frac{u_M - u_m}{t - \tau_m}\right), \ \tilde{\sigma}_m = \left(\frac{M-m-1}{M-m}\right)\bar{\sigma}^2$$

This sampler is called the modified Brownian bridge[4].

Although it is possible to compute the likelihood directly from (B.20), it is time-consuming. Suppose we have data generated, on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, by the process

$$dX = f(X)dt + g(X)dB^{\mathbb{P}} \tag{B.21}$$

where B is a $d-$dimensional Brownian motion under the probability $\mathbb{P}$. Suppose we want to change the drift to the process by including $\gamma = \tilde{\mu}(X) - f(X)$ so that the drift becomes $\tilde{\mu}(X)$. Provided that $\gamma_t(X_t)$ is adapted to B_t and there is an adapted solution u to the equation

$$u(X) = g(X)^{-1}(\tilde{\mu}(X) - f(X)) \tag{B.22}$$

then the process can be rewritten as

$$dX = \tilde{\mu}(X)dt + g(X)[u(X)dt + dB_t^{\mathbb{P}}]$$

under $\mathbb{P}$. The process will also satisfy

$$d\tilde{X} = \tilde{\mu}(\tilde{X})dt + g(\tilde{X})d\tilde{B}_t^{\mathbb{Q}} \tag{B.23}$$

Assuming weak uniqueness, the solution of the process (B.21) as the same distribution than the process in (B.23). Girsanov's theorem tells us that the Radon-Nykodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$ is

$$\begin{aligned}\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} &= M_t \\ &= exp\left\{-\sum_{i=1}^{d}\int_0^t u^{(i)}(X_s)dB_s^{(i)} - \frac{1}{2}\int_0^t \|u(X_s)\|^2 ds\right\}\end{aligned}$$

or written differently, under P

$$dM_t = M_t\left(\sum_{i=1}^{d} -u^{(i)}(X_s)dB_s^{(i)}\right)$$

[4]It is named after Durham and Gallant (2002).

or under Q,

$$dM_t = M_t \left(\sum_{i=1}^{d} u^{(i)}(X_s) d\tilde{B}_s^{(i)} \right)$$

with the initial condition that $M_s = 1$ and where $u^{(i)}(X_s)$ refers the the i^{th} coordinate of (B.22). Thus one can obtain the continuous-time expression

$$p(x_t, t; x_s, s) = \int p(x_t, t; u, \tau_{M-1}) \rho_{M-1}(u) dQ_{M-1}(u), \tag{B.24}$$

where Q_{M-1} is the probability measure induced by $\tilde{X}_{\tau_{M-1}}$. The integral is computed by generating samples $\{(u_{k,M-1}, r_{k,M-1})\}$ from the joint process $(\tilde{X}_{M-1}^{(M)}, M_{M-1}^{(M)})$ using the Euler-Maruyama scheme,

$$p^{(M,K)}(x_t, t; x_s, s, \theta) = \frac{1}{K} \sum_{k=1}^{K} p^{(1)}(x_t, t; u_{k,M-1}, \tau_{M-1}) r_{k,M-1}.$$

Durham and Gallant (2002) found that it is more stable to base the Euler-Maruyama scheme for M on

$$d(\log(M)) = -\frac{1}{2} \sum_{k=1}^{d} (u^{(i)}(\tilde{X}))^2 dt + \sum_{k=1}^{d} u^{(i)}(\tilde{X}) d\tilde{B} \tag{B.25}$$

Finally, I will compute the simulated log-likelihood $l_n^{(M,K)}(\theta) = \sum_{i=1}^{n} \log p^{(M,K)}(X_i, t_i; X_{i-1}, t_{i-1}, \theta)$.

B.3 Example of the estimation method with a DGP

In order to run the example, one can consider the stochastic Lotka-Volterra model,

$$\begin{aligned} d\omega_t &= \omega_t \left((\phi(\lambda_t) - \alpha + \sigma_1^2(\omega_t, \lambda_t)) dt + \sigma_1(\omega_t, \lambda_t) dB_t^1 + \sigma_2(\omega_t, \lambda_t) dB_t^2 \right) \\ d\lambda_t &= \lambda_t \left(\left[\frac{1 - \omega_t}{\nu} - (\alpha + \beta + \delta) + \sigma_1^2(\omega_t, \lambda_t) \right] dt + \sigma_1(\omega_t, \lambda_t) dB_t^1 \right). \end{aligned} \tag{B.26}$$

When assuming $\sigma_1(\omega_t, \lambda_t) = \sigma_1$ and $\sigma_2(\omega_t, \lambda_t) = \sigma_2$, the system (B.26) written in the form of (B.17), one can identify

- $X_t = \begin{pmatrix} \omega_t \\ \lambda_t \end{pmatrix}$,
- $B_t = \begin{pmatrix} B_t^1 \\ B_t^2 \end{pmatrix}$,
- $f \begin{pmatrix} \omega_t \\ \lambda_t \end{pmatrix} = \begin{pmatrix} \omega_t(\phi(\lambda_t) - \alpha + \sigma_1^2) \\ \lambda_t \left(\left[\frac{1-\omega_t}{\nu} - (\alpha + \beta + \delta) + \sigma_1^2 \right] \right) \end{pmatrix}$,

- $g\begin{pmatrix}\omega_t\\ \lambda_t\end{pmatrix} = \begin{pmatrix}\omega_t\sigma_1 & \omega_t\sigma_2\\ \lambda_t\sigma_1 & 0\end{pmatrix}.$

To test the estimation, the following process will be generate

$$d\begin{pmatrix}\omega_t\\ \lambda_t\end{pmatrix} = \begin{pmatrix}\omega_t(\phi_1\lambda_t - \phi_0)\\ \lambda_t\left(\frac{1-\omega_t}{\psi_1} - \psi_0\right)\end{pmatrix} dt + \begin{pmatrix}\sigma_1\omega_t & \sigma_2\omega_t\\ \sigma_1\lambda_t & 0\end{pmatrix} d\begin{pmatrix}B_t^1\\ B_t^2\end{pmatrix}, \tag{B.27}$$

where the parameters will be

- $\phi_0 = 0.22$
- $\phi_1 = 0.25$
- $\psi_0 = 0.29$
- $\psi_1 = 1.2$
- $\sigma_1 = 0.005$
- $\sigma_2 = 0.005$

The simulation is made for 50 years and a sample is build by picking data at every quarter, to reproduce common macroeconomic data's frequency.

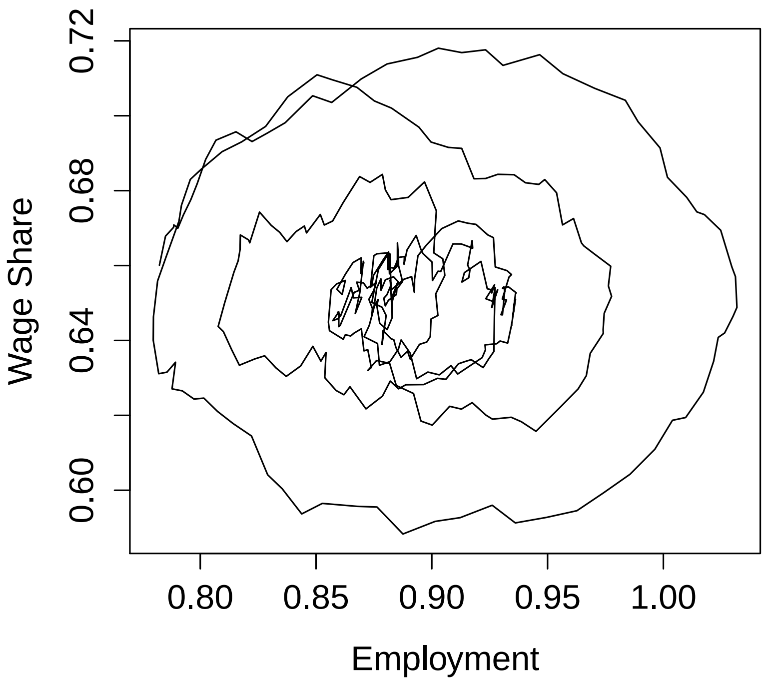
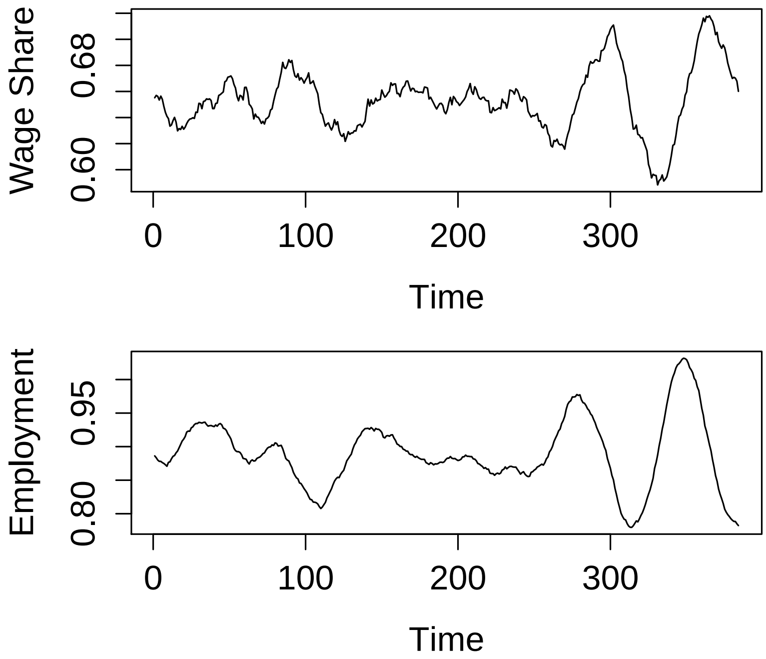


FIGURE B.1: Simulation: The phase portrait of the system B.27

The starting value of the simulation are $(\omega_0, \lambda_0) = (0.651, 0.890)$.



B.3.1 Test of the Estimation

For the estimation, one can use $M = 8$ and $K = 16$.[5] The starting values are chosen randomly.

Parameter	Starting Value	Estimated Value (standard deviation)	True Parameter
ϕ_0	0.12	0.216873654 (0.0170)	0.22
ϕ_1	0.8	0.243865676 (0.0191)	0.25
ψ_0	0.25	0.289664962 (0.0060)	0.29
ψ_1	1	1.201835107 (0.0250)	1.2
σ_1	0.05	0.004519791 (0.0002)	0.005
σ_2	0.05	-0.009546113 (0.0003)	0.01

Table B.1: Results for the first estimation. $M = 8$, $K = 16$.

Regarding the results of table B.1, one can note that we cannot rely on the sign of the sigmas since the Brownian motion is symmetric[6] and the model specification is linear with respect to the Brownian motion.

[5] These parameters are chosen to make the tradeoff between time of computation and accuracy of the results.

[6] Indeed, if W_t is a Brownian motion, $-W_t$ is also a Brownian motion.

B.4 Numerical Test of the Inference of the Short Term Phillips Curve

This appendix aims to show how a linear regression techniques to estimate a continuous-time short term linear Phillips curve of the form

$$\frac{\dot{W}}{W} = \Phi(\lambda)$$

can lead to spurious results. For that purpose, we generate data for the employment rate, λ. Suppose that λ is generated according to an autoregressive process with a lag one, so that

$$\lambda_t = 0.96 * (1 - 0.5^{1/(N/T/4)}) + 0.5^{1/(N/T/4)} \lambda_{t-1/N} + \varepsilon_t,$$

where $\varepsilon_t \sim \mathcal{N}(0, T/N)$. The parameters are chosen so that the mean for λ is 0.96 and the correlation of λ_t and $\lambda_{t+1/4}$ is 0.5. $N = 50 \times 10000$ is the number of subperiods between 0 and $T = 50$. Using that λ, the wages will be simulated using the Euler-Maruyama scheme of the stochastic differential process

$$dW_t = W_t \left((\phi(\lambda_t))dt + \sigma dB_t \right).$$

With $W_0 = 100$, and $\sigma = 0.01$. The $\Phi(.)$ function is suppose to be linear so that

$$\begin{aligned} \Phi(\lambda) &= \phi_0 \times \lambda + \phi_1 \\ &= 0.89 \times \lambda - 0.82 \end{aligned}$$

Two samples will be created by talking the the value that correspond to one quarter for both λ and W. The log −return of the quarterly timeserie of W is computed, its scatterplot with the quarterly timeserie of λ. If one uses linear regression techniques on

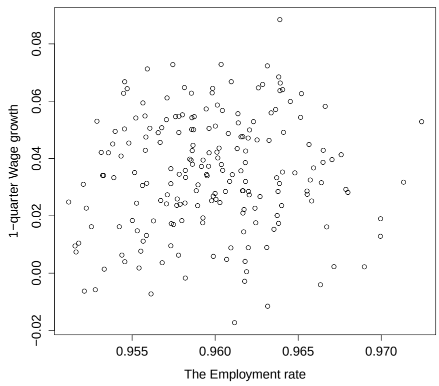


FIGURE B.2: Simulation: The employment rate versus the one-quarter wage growth.

the quarterly data to estimate the model of the DGP given above, one finds

$$\hat{\phi}_0 = 0.371;\ \hat{\phi}_1 = -0.3214.$$

One can conclude that, in this example, the results are at odds with the parameters used for the DGP.

B.5 Estimation and Backtesting with Mohun et al(2006)'s Data

This appendix resumes the backtesting and the estimation of the models with the time-serie for the US of ω and λ constructed in Mohun and Veneziani (2006).[7] The same short term Phillips curve are tested,

$$\begin{aligned} \phi(\lambda) &= \phi_0 + \phi_1 \lambda, && \text{(B.28)} \\ &= \phi_0 + \frac{\phi_1}{(1-\lambda)}, && \text{(B.29)} \\ &= \phi_0 + \frac{\phi_2}{(1-\lambda)^2}. && \text{(B.30)} \end{aligned}$$

on the same time span. Interestingly the order of the AIC criterion remains the same,

	Leontief	CES
Short term Phillips curve (3.6)	−1983.028	−2023.347
Short term Phillips curve (3.7)	−1983.837	−2027.587
Short term Phillips curve (3.8)	−1985.257	−2029.904

TABLE B.2: The AIC values of the Leontief and the CES models.

The Parameter Estimation for the Leontief

	ϕ_1	ϕ_0	ψ_0	ψ_1	σ_1	σ_2
PC (3.6)	0.8012 (0.1756)	0.7545 (0.1654)	4.2079 (0.6446)	0.0521 (0.0080)	0.0078 (0.0003)	0.0234 (0.0010)
PC (3.7)	0.0025 (0.0005)	0.04758 (0.0102)	4.1409 (0.6247)	0.0531 (0.0080)	0.0078 (0.0003)	0.0233 (0.0010)
PC (3.8)	$5.827e-05$ ($1.599e{-}05$)	0.0219 (0.0065)	4.1242 (0.6266)	0.0531 (0.0081)	0.0078 (0.0003)	0.0234 (0.0101)

TABLE B.3: The parameter estimates (the standard deviation).

[7] The data taken in this section are named 'Dataset 1' in Mohun and Veneziani (2006). One can refer the reader to the paper for an extensive discussion on the construction of the dataset.

The Parameter Estimation for the CES

	ϕ_1	ϕ_0	C_π	ψ_1	η	σ_1	σ_2
PC (3.6)	0.8090 (0.1511)	0.7620 (0.1423)	0.2509 (0.0385)	0.0078 (0.0513)	20.5999 (2.6408)	0.0083 (0.0004)	0.0201 (0.0009)
PC (3.7)	0.0026 (0.0004)	0.0491 (0.0088)	0.2578 (0.0389)	0.0527 (0.0079)	20.1931 (2.5745)	0.0083 (0.0004)	0.01996 (0.0009)
PC (3.8)	$6.059e-05$ ($1.392e{-}05$)	0.0229 (0.0057)	0.256 (0.039)	0.0524 (0.008)	20.040 (2.5637)	0.0837 (0.0004)	0.0199 (0.0009)

TABLE B.4: The parameters estimate (the standard deviation).

B.5.1 Backtesting

	Leontief		CES	
	ω	λ	ω	λ
h = 1	**0.9023222**	1.2670517	**0.9034798**	1.2507662
h = 2	**0.8443463**	1.1187807	**0.8474636**	1.1129887
h = 3	**0.8777915**	1.0130869	**0.8768499**	**0.9706326**
h = 4	**0.8767185**	**0.9474305**	**0.8763229**	**0.8965066**
h = 5	**0.9004310**	**0.9358985**	**0.8839289**	**0.8719911**
h = 6	**0.9089183**	**0.9487362**	**0.8804783**	**0.8736256**
h = 7	**0.9221140**	**0.9747208**	**0.8753814**	**0.8894193**
h = 8	**0.9493549**	**0.9911088**	**0.8778018**	**0.8907999**

TABLE B.5: Relative performance of the model against a VAR(10) with the short term Phillips curve (3.6).

	Leontief		CES	
	ω	λ	ω	λ
h = 1	**0.8967319**	1.2756787	**0.8981564**	1.3160756
h = 2	**0.8310902**	1.1223907	**0.8377516**	1.1756384
h = 3	**0.8527786**	1.0097811	**0.8649906**	1.0178919
h = 4	**0.8475473**	**0.9388411**	**0.8638524**	**0.9318095**
h = 5	**0.8617221**	**0.9235656**	**0.8825049**	**0.9046815**
h = 6	**0.8545877**	**0.9257241**	**0.8686369**	**0.9070944**
h = 7	**0.8444716**	**0.9491274**	**0.8571461**	**0.9248434**
h = 8	**0.8619737**	**0.9615577**	**0.8758972**	**0.9258368**

TABLE B.6: Relative performance of the model against a VAR(10) with the short term Phillips curve (3.7).

Interestingly, the conclusions are similar, perhaps more optimistic, for those dataset than the data consider in the paper.

	Leontief		CES	
	ω	λ	ω	λ
h = 1	**0.8999064**	1.2637668	**0.9006265**	1.2761530
h = 2	**0.8426832**	1.1068832	**0.8496162**	1.1293967
h = 3	**0.8677856**	**0.9947445**	**0.8800742**	**0.9774874**
h = 4	**0.8584810**	**0.9230021**	**0.8776383**	**0.8975758**
h = 5	**0.8711563**	**0.9084126**	**0.8902298**	**0.8739785**
h = 6	**0.8513840**	**0.9090925**	**0.8736040**	**0.8754481**
h = 7	**0.8401227**	**0.9329282**	**0.8616404**	**0.8949710**
h = 8	**0.8668517**	**0.9414028**	**0.8897422**	**0.9034567**

Table B.7: Relative performance of the model against a VAR(10) with the short term Phillips curve (3.8).

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Appendix C

Appendix Chapter 4

C.1 Getting the reduced form of the system

We assume that the productive sector is endowed with a CES technology so that

$$Y = C\left[bK^{-\eta} + (1-b)(e^{a_l t}L)^{-\eta}\right]^{-\frac{1}{\eta}}. \tag{C.1}$$

Additionally, we make the assumption that wages are set at marginal rate of productivity, so that:

$$\frac{\partial Y}{\partial L} = \text{w}.$$

For simplicity, we define $L^e := e^{a_l t}L$, so that the following relationship holds

$$\frac{\partial Y}{\partial L^e} = \frac{\partial Y}{\partial L}e^{-a_l t}. \tag{C.2}$$

By using equations (C.1) and (C.2)

$$\frac{\partial Y}{\partial L^e} = \frac{(1-b)}{C^\eta}\left(\frac{Y}{L^e}\right)^{1+\eta}.$$

By taking the derivative of (C.1) and using (C.2)

$$\begin{aligned}
&\left(\frac{\omega}{1-b}\right)^{\frac{1}{\eta}} C = \frac{Y}{L^e} \\
\Leftrightarrow \quad &\left(\frac{\omega}{1-b}\right)^{\frac{1}{\eta}} Ce^{a_l t} = \frac{Y}{L}
\end{aligned} \tag{C.3}$$

with ω, the share of total real wages ($W := \mathrm{w}L$) in the production:

$$\omega := \frac{\mathrm{w}L}{Y}.$$

Let $a := Y/L$ be the labor productivity, one has $\omega = \mathrm{w}/a$. The growth rate of the wage share is given by

$$\frac{\dot{\omega}}{\omega} = \frac{\dot{\mathrm{w}}}{\mathrm{w}} - \frac{\dot{a}}{a}.$$

Using equation (C.3), one gets the following growth rate for labor productivity

$$\frac{\dot{a}}{a} = \frac{1}{\eta}\frac{\dot{\omega}}{\omega} + a_l.$$

Suppose that the growth rate of wages is given by a short-term Phillips Curve

$$\frac{\dot{\mathrm{w}}}{\mathrm{w}} = \Phi(\lambda).$$

The dynamic of the wage share is given by

$$\boxed{\frac{\dot{\omega}}{\omega} = \left(\frac{\eta}{1+\eta}\right)\left[\Phi(\lambda) - a_l\right]}\,.$$

The population grows according to

$$\frac{\dot{N}}{N} = \beta \geq 0.$$

The employment rate is defined by $\lambda := \frac{L}{N}$, while the capital-output ratio is given by $\nu := \frac{K}{Y}$. Hence, the employment rate dynamic

$$\begin{aligned}\frac{\dot{\lambda}}{\lambda} &= \frac{\dot{L}}{L} - \frac{\dot{N}}{N} \\ &= \frac{\dot{K}}{K} - \frac{\dot{a}}{a} - \frac{\dot{\nu}}{\nu} - \frac{\dot{N}}{N}.\end{aligned}$$

The profit share in the production is given by

$$\pi := 1 - \omega - rd,$$

where r is the short-term interest rate set by the central bank, and paid by producers, while d is the ratio of real debt-to-production $\left(\text{i.e } \frac{D}{Y}\right)$. The capital accumulation is given by

$$\dot{K} = \kappa(\pi)Y - \delta K,$$
$$\frac{\dot{K}}{K} = \frac{\kappa(\pi)}{\nu} - \delta$$

where δ is the depreciation rate of capital, $\kappa(\pi)$ is a function of the profit share, and ν is the time-varying capital-to-output ratio. From the expression of $\frac{\partial Y}{\partial K}$ and knowing that Y is homogeneous of degree one, we obtain

$$\nu = \left(\frac{1-\omega}{b}\right)^{-\frac{1}{\eta}} \frac{1}{C}.$$

Its growth rate is given by

$$\frac{\dot{\nu}}{\nu} = \frac{\dot{\omega}}{(1-\omega)\eta}.$$

Therefore, the growth rate of employment is

$$\boxed{\frac{\dot{\lambda}}{\lambda} = \kappa(\pi)C\left(\frac{1-\omega}{b}\right)^{1/\eta} - \delta - a_l - \beta - \frac{1}{\eta(1-\omega)}\frac{\dot{\omega}}{\omega}}.$$

The debt dynamic is the difference between investment and the profits made by the corporate sector, in other words

$$\dot{D} = \kappa(\pi)Y - (\pi)Y.$$

The growth rate of production is given by

$$g := \frac{\dot{Y}}{Y} = \kappa(\pi)C\left(\frac{1-\omega}{b}\right)^{1/\eta} - \delta - \frac{\dot{\omega}}{(1-\omega)\eta}.$$

Thus, the ratio of real debt on production is

$$\frac{\dot{d}}{d} = \frac{\dot{D}}{D} - \frac{\dot{Y}}{Y} = \frac{\kappa(\pi)-\pi}{d} - \kappa(\pi)C\left(\frac{1-\omega}{b}\right)^{1/\eta} + \delta + \frac{\dot{\omega}}{(1-\omega)\eta}.$$

Hence, its dynamic is

$$\boxed{\dot{d} = d\left\{r - \kappa(\pi)C\left(\frac{1-\omega}{b}\right)^{1/\eta} + \delta + \frac{\dot{\omega}}{(1-\omega)\eta}\right\} + \kappa(\pi) - (1-\omega)}.$$

To wrap up, and for the sake of clarity, the three-dimensional system is

$$\begin{cases} \dot{\omega} & = \omega \left[\left(\frac{\eta}{1+\eta} \right) [\Phi(\lambda) - a_l] \right] \\ \dot{\lambda} & = \lambda \left[\kappa(\pi) C \left(\frac{1-\omega}{b} \right)^{1/\eta} - \delta - a_l - \beta - \frac{1}{\eta(1-\omega)} \frac{\dot{\omega}}{\omega} \right] \\ \dot{d} & = d \left[r - \kappa(\pi) C \left(\frac{1-\omega}{b} \right)^{1/\eta} + \delta + \frac{\dot{\omega}}{(1-\omega)\eta} \right] + \kappa(\pi) - (1-\omega) \end{cases}$$

C.2 Parameter values

The calibration is almost entirely borrowed from Keen (2013). The time frequency between t and $t+1$ is considered to be one year. The same generalized exponential

Variable or parameter	Description	Value
a_l	Rate of the labor productivity growth	2%
β	Rate of population growth	1%
δ	Depreciation rate of capital	1%
$f(\lambda, 0.95, 0, 0.05, -0.01)$	Parameters for the nonlinear Phillips curve	
$f(\pi, 0.05, 0.05, 1.75, 0)$	Parameters for the investment function	
r	Interest rate paid by the productive sector	4%
η	Control the elasticity of substitution	$\approx 0; 1; +\infty$
C	The factor productivity	1/3
b	The share of capital in the production function	1 - 0.865 = 0.135

Table C.1: Calibration for the numerical estimations.

function is used for both the relationship between investment as a share of output, and the short term Phillips Curve

$$f(x, x_{\text{val}}, y_{\text{val}}, s, \min) = (y_{\text{val}} - \min) e^{s/(y_{\text{val}}) \times (x - x_{\text{val}})} + \min .$$

Figure C.1 displays the behaviors of the phenomenological functions using the calibration given by Table C.1.

C.3 Numerical Results for the Stability of Equilibrium

Value of η	Eigen. 1.	Eigen. 2.	Eigen. 3.
$+\infty$	-0.022+1.33i	-0.022-1.33i	-0.030
100	-0.075+1.32i	-0.075-1.32i	-0.030
0	-9.69	-0.162	-0.037
$-1/2$	-0.0498+0.0233i	-0.0498-0.023i	-21.575

Table C.2: The numerical eigenvalues of all the models at their *good* equilibrium point $(\omega_1, \lambda_1, d_1)$.

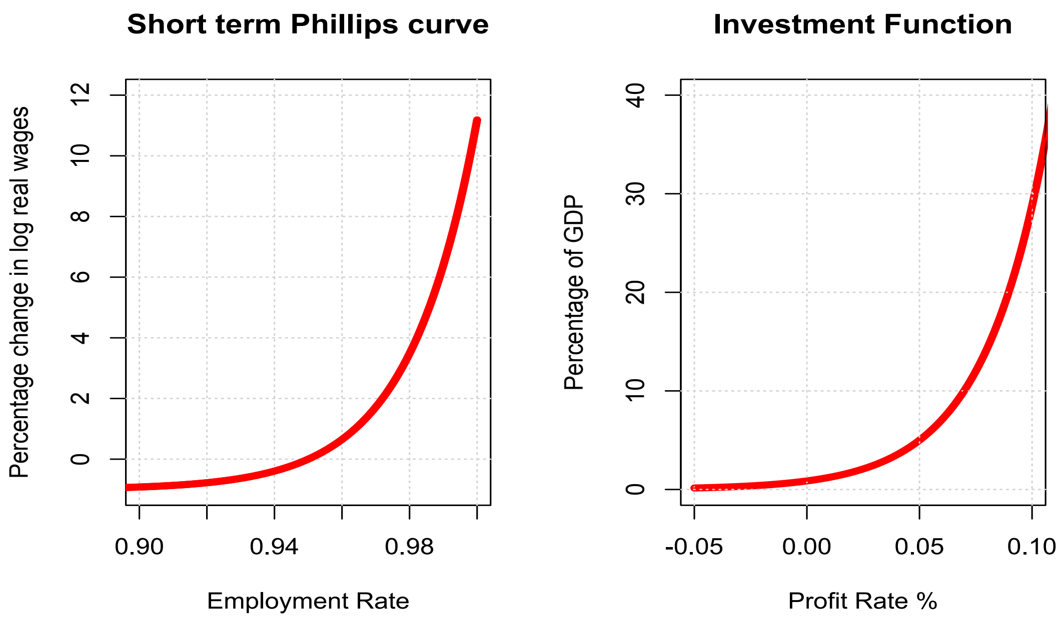

FIGURE C.1: Phenomenological functions behaviors according to the parameters in Table C.1.

Table C.2 displays the numerical eigenvalues of the Jacobian at the *good* equilibrium point, where $(\omega_1, \lambda_1, d_1)$ are all finite. Remember, to be locally stable, the Jacobian matrix at this equilibrium point has to be negative definite. This would mean that the eigenvalues are all non-positive. In this exercise, they all show local stability at the *good* equilibrium value.[1] For the sake of completeness, Tables C.3 and C.4 show respectively

Value of η	Eigen. 1.	Eigen. 2.	Eigen. 3.
$+\infty$	6.8e+10	-6.88+10	-0.03
100	0.05	-0.04	-0.03
0	0.074	-0.012	-0.003
$-1/2$	0.02	0.02	1.69e-08

TABLE C.3: The numerical eigenvalues of all the models at their *obvious* equilibrium point $(\bar{\omega}_3, \bar{\lambda}_3, \bar{d}_3)$.

Value of η	Eigen. 1.	Eigen. 2.	Eigen. 3.
$+\infty$	-	-	-
100	0.05	-0.04	-0.03
0	1.27	-0.52	-0.002
$-1/2$	-0.88	0.021	-0.0016

TABLE C.4: The numerical eigenvalues of all the models at the *slavery* equilibrium point $(\bar{\omega}_2, \bar{\lambda}_2, \bar{d}_2)$.

[1] In the simulation, the value 0 would lead to numerical errors, therefore we choose the value 0.1 as the lowest value that does not show numerical error.

the eigenvalues of the *obvious* and the *slavery* equilibria. We observe that, as expected, non of these trials display a local stability for that equilibrium point. The eigenvalues

Value of η	Eigen. 1.	Eigen. 2.	Eigen. 3.
$+\infty$	-0.03	-0.04	-0.05
100	-0.03	-0.04	-0.05
0	-0.003	-0.01	-0.05
$-1/2$	0.03	0.02	-0.05

TABLE C.5: The numerical eigenvalues of all the models at their *bad* equilibrium point $(\omega_3, \lambda_3, d_3)$.

of the *bad* equilibrium are shown in Table C.5 and confirm that, whenever $\eta \in]-1;0[$, the only stable equilibrium that may be asymptotically globally stable is the *good*.

C.4 Equilibria for an Affine Investment Function

For the sake of completeness, what follows derives theoretically the equilibrium points under the assumption of an affine investment function.

On the one hand, one can easily find the case $\omega_1 \neq 0$, one has

$$\bar{\lambda}_1 = \Phi^{-1}(a_l).$$

For what follows, we will consider that $\kappa(.)$ is an affine function, i.e $\kappa(x) = K_0 + K_1 x$. By using the third equation of the system (4.2), one has

$$\begin{aligned}
& -\bar{d}_1(\beta + a_l) + \kappa(\pi) - \pi = 0 \\
\Leftrightarrow \quad & -K_0 - K_1\pi + \pi + \bar{d}_1(\beta + a_l) = 0 \\
\Leftrightarrow \quad & (1 - K_1)\pi - K_0 + \bar{d}_1(\beta + a_l) = 0
\end{aligned}$$

where $\pi := 1 - \bar{\omega}_1 - r\bar{d}_1$ and K_0 and K_1 are constants. Thus,

$$\begin{aligned}
(1-\bar{\omega}_1)(1-K_1) + \bar{d}_1(\beta + a_l - r(1-K_1)) - K_0 &= 0 \\
\frac{K_0 + (1-\bar{\omega}_1)(K_1 - 1)}{\beta + a_l + r(K-1)} &= \bar{d}_1.
\end{aligned}$$

By adding the last equality of the second equation of the system (4.2), one has that $\bar{\omega}_1$ is such that satisfies with

$$\begin{aligned}
&(\delta + a_l + \beta)\, b^{1/\eta} \left[\beta + a_l + r(K_1 - 1)\right] \\
&\qquad = \left[K_0(\beta + a_l - r) + K_1(1-\bar{\omega}_1)(\beta + a_l)\right] C(1-\bar{\omega}_1)^{1/\eta}
\end{aligned}$$

Consider another non-trivial equilibrium taking the form of

$$(\bar{\omega}_2 = 0, \quad \bar{\lambda}_2 \neq 0, \quad \bar{d}_2 \neq 0)$$

After plugging the first equation of system (4.2) into the second and third equations, the evaluation of the mentioned equilibrium into the second equation of (4.2), yields

$$\kappa(1-rd)C\left(\frac{1}{b}\right)^{1/\eta} - \delta - a_l - \beta - \left(\frac{1}{1+\eta}\right)[\Phi(\lambda) - a_l] = 0$$

Likewise, from the third equation in (4.2) it follows that:

$$d\left[r - \kappa(1-rd)C\left(\frac{1}{b}\right)^{1/\eta} + \delta\right] + \kappa(1-rd) - 1 = 0$$

Solving the of these equations for $\kappa(1-rd)C\left(\frac{1}{b}\right)^{1/\eta}$ and replacing it in the second equation, produces

$$d\left[r - a_l - \beta - \left(\frac{1}{1+\eta}\right)[\Phi(\lambda) - a_l]\right] + \kappa(1-rd) - 1 = 0$$

In the same line as before, consider the case where $\kappa(x) = K_0 + K_1 x$, for which the expression above would become

$$d\left[r - a_l - \beta - \left(\frac{1}{1+\eta}\right)[\Phi(\lambda) - a_l] - rK_1\right] + K_0 + K_1 - 1 = 0$$
$$\iff \quad \bar{d}_2 = \frac{1 - K_0 - K_1}{r(1-K_1) - a_l - \beta - \left(\frac{1}{1+\eta}\right)[\Phi(\lambda) - a_l]}$$

It can be seen by plugging this into the second equation of system (4.2), that the equilibrium employment will be any $\bar{\lambda}_2$ satisfying:

$$\frac{rK_0 + (K_0 + K_1)\xi}{r(1-K_1) + \xi}\,\frac{C}{b^{1/\eta}} - \delta + \xi = 0$$

with ξ being itself an expression in terms of $\bar{\lambda}_2$

$$\xi := -a_l - \beta - \left(\frac{1}{1+\eta}\right)\left[\Phi(\bar{\lambda}_2) - a_l\right].$$

C.5 Existence of the *slavery* equilibrium

Let us take the last two equations of the main system when $\omega \to 0$.

$$\dot{\lambda} \simeq \lambda \left[\kappa (1 - rd) C b^{-\frac{1}{\eta}} - \frac{\Phi(\lambda) - a_l}{1 + \eta} - (\delta + a_l + \beta) \right] \quad (C.4)$$

$$\dot{d} \simeq d \left(r - \kappa (1 - rd) C b^{-\frac{1}{\eta}} + \delta + \omega \frac{\Phi(\lambda) - a_l}{1 + \eta} \right) + \kappa (1 - rd) - 1 \quad (C.5)$$

At the equilibrium, whenever $\lambda > 0$, one finds, by injecting C.4 into C.5:

$$0 \simeq d \left(r - \kappa (1 - rd) C b^{-\frac{1}{\eta}} + \delta + \omega \left(\kappa (1 - rd) C b^{-\frac{1}{\eta}} - \delta - a_l - \beta \right) \right) + \kappa (1 - rd) - 1$$

As κ is bounded, one can neglect the term in ω. Defining $s = C b^{-\frac{1}{\eta}}$, $e = r + \delta > 0$ and $z(d) = -\kappa (1 - rd) \in [-1; 0]$ one thus obtains

$$d (s z(d) + e) = 1 + z(d) \in [0; 1]$$

If $s < e$, the left hand side is a continuous and non-negative function of d which passes through the origin and is equivalent to $d (e - s\kappa_0)$ at $+\infty$. Hence, the equation has a solution and $s < e$ is a sufficient condition for the existence of a *slavery* equilibrium.

For $\eta < 0$, s converges decreasingly towards 0 when η tends to 0, so there exists an interval $]-\eta_{min}; 0[$ within which the existence of the equilibrium is insured.

Note that for $\eta > 0$, as $b < 1$, s decreases with η and converges towards C, so that if there is a substitutability η_0 such that $s_{\eta_0} < e = r + \delta$, the existence of the slavery equilibrium is insured below this substitutability (for $\eta > \eta_0$). That being said, for the benchmark specification, this inequality does not hold. One can then derive a less restrictive sufficient condition: $e - s\kappa_0 > 0$. Finally, if $\kappa_0 < b^{\frac{1}{\eta}}(r+\delta)/C$, a slavery equilibrium exists (with an equilibrium value for the debt potentially very high). For our benchmark specification with $\eta = 1$ (resp. $\eta = \infty$, resp. $\eta = -0.5$), it suffices that $\kappa_0 < 0.02$ (resp. $\kappa_0 < 0.15$, resp. $\kappa_0 < 8.2$) for the equilibrium to exist.

C.6 Getting the reduced form of the system with a Cobb-Douglas production function

We assume that the productive sector is endowed with a constant return to scale Cobb-Douglas technology so that

$$Y = C\left[K^b(e^{a_l t}L)^{(1-b)}\right]. \tag{C.6}$$

Additionally, we make, as previously, the assumption that wages are set at marginal rate of productivity, so that:

$$\frac{\partial Y}{\partial L} = \mathrm{w}.$$

For simplicity, we define $L^e := e^{a_l t}L$, so that the following relationship holds

$$\frac{\partial Y}{\partial L^e} = \frac{\partial Y}{\partial L}e^{-a_l t}. \tag{C.7}$$

By using equations (C.6) and (C.7)

$$\frac{\partial Y}{\partial L^e} = (1-b)\left(\frac{Y}{L^e}\right).$$

By taking the derivative of (C.1) and using (C.2)

$$\omega = (1-b) \tag{C.8}$$

with ω, the share of total real wages ($W := \mathrm{w}L$) in the production:

$$\omega := \frac{\mathrm{w}L}{Y}.$$

Let $a := Y/L$ be the labor productivity, one has $\omega = \mathrm{w}/a$. The growth rate of the wage share is given by

$$\frac{\dot\omega}{\omega} = \frac{\dot{\mathrm{w}}}{\mathrm{w}} - \frac{\dot a}{a}.$$

Using equation (C.8), one gets the following growth rate for labor productivity

$$\frac{\dot a}{a} = \frac{\dot{\mathrm{w}}}{\mathrm{w}}.$$

Suppose that the growth rate of wages is given by a short-term Phillips Curve

$$\frac{\dot{\mathrm{w}}}{\mathrm{w}} = \Phi(\lambda).$$

The dynamic of the wage share is given by

$$\boxed{\frac{\dot{\omega}}{\omega} = 0}\,.$$

The population grows according to

$$\frac{\dot{N}}{N} = \beta \geq 0.$$

The employment rate is defined by $\lambda := \frac{L}{N}$, while the capital-output ratio is given by $\nu := \frac{K}{Y}$. Hence, the employment rate dynamic

$$\begin{aligned}\frac{\dot{\lambda}}{\lambda} &= \frac{\dot{L}}{L} - \frac{\dot{N}}{N} \\ &= \frac{\dot{K}}{K} - \frac{\dot{a}}{a} - \frac{\dot{\nu}}{\nu} - \frac{\dot{N}}{N}.\end{aligned}$$

The profit share in the production is given by

$$\begin{aligned}\pi &:= 1 - \omega - rd, \\ &= b - rd\end{aligned}$$

where r is the short-term interest rate set by the central bank, and paid by producers, while d is the ratio of real debt-to-production $\left(\text{i.e } \frac{D}{Y}\right)$. The capital accumulation is given by

$$\begin{aligned}\dot{K} &= \kappa(\pi)Y - \delta K, \\ \frac{\dot{K}}{K} &= \frac{\kappa(\pi)}{\nu} - \delta\end{aligned}$$

where δ is the depreciation rate of capital, $\kappa(\pi)$ is a function of the profit share, and ν is the time-varying capital-to-output ratio. By dividing equation C.6, by the output, Y, we are able to obtain

$$\nu = \left(\frac{1}{C}\right)^{1/b} \left(\frac{a}{e^{a_l t}}\right)^{\frac{1-b}{b}}.$$

Its growth rate is given by

$$\frac{\dot{\nu}}{\nu} = \frac{1-b}{b}\left(\Phi(\lambda) - a_l\right).$$

Therefore, the growth rate of employment is

$$\boxed{\frac{\dot{\lambda}}{\lambda} = \frac{\kappa(\pi)}{\nu} - \delta - \Phi(\lambda) - \beta - \frac{1-b}{b}\left(\Phi(\lambda) - a_l\right)}\,.$$

The debt dynamic is the difference between investment and the profits made by the corporate sector, in other words

$$\dot{D} = \kappa(\pi)Y - (\pi)Y.$$

The growth rate of production is given by

$$g := \frac{\dot{Y}}{Y} = \frac{\kappa(\pi)}{\nu} - \delta - \frac{1-b}{b}\left(\Phi(\lambda) - a_l\right).$$

Thus, the ratio of real debt on production is

$$\frac{\dot{d}}{d} = \frac{\dot{D}}{D} - \frac{\dot{Y}}{Y} = \frac{\kappa(\pi) - \pi}{d} - \frac{\kappa(\pi)}{\nu} + \delta + \frac{1-b}{b}\left(\Phi(\lambda) - a_l\right).$$

Hence, its dynamic is

$$\boxed{\dot{d} = d\left\{r - \frac{\kappa(\pi)}{\nu} + \delta + \frac{1-b}{b}\left(\Phi(\lambda) - a_l\right)\right\} + \kappa(\pi) - b}\,.$$

To wrap up, and for the sake of clarity, the three-dimensional system is

$$\begin{cases} \dot{\nu} & = \nu\left(\frac{1-b}{b}\left[\Phi(\lambda) - a_l\right]\right) \\ \dot{\lambda} & = \lambda\left[\frac{\kappa(\pi)}{\nu} - \delta - \Phi(\lambda) - \beta - \frac{1-b}{b}\left(\Phi(\lambda) - a_l\right)\right] \\ \dot{d} & = d\left\{r - \frac{\kappa(\pi)}{\nu} + \delta + \frac{1-b}{b}\left(\Phi(\lambda) - a_l\right)\right\} + \kappa(\pi) - b \end{cases}$$

References

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Appendix D

Appendix Chapter 5

The Appendices are ordered as follows: i) the description of the empirical data we constructed for the model estimation; ii) the calibration of the temperature dynamics; iii) the estimation of the demographic scenario; iv) the calibration of the CO_2 dynamics; v) the estimation of the macroeconomics dynamics; and vi) additionnal simulations.

D.1 Data Collection

To estimate the parameters of the model at the global level, we collected historical data representing more than 80% of the world's GDP. Since global aggregate data are not available through open source, we collected the data at country level.

The final sample is based on the data of 36 countries. It includes 18 members of G20,[1] 16 members of OECD that are not individually part of the G20,[2] and two non-OECD countries.[3] The combined GDP of the listed countries accounted for 87% of world GDP in 2014.[4] At the time we were collecting our data, there was a lack of data on capital stock and wages after 2011, so we limited our time frame to the period 1991-2011. A detailed description of the methodology used to construct the database is given below.

[1]Argentina, Australia, Brazil, Canada, China, France, Germany, India, Indonesia, Italy, Japan, South Korea, Mexico, Russia, South Africa, Turkey, the United Kingdom, and the United States.

[2]Austria, Belgium, Czech Republic, Denmark, Finland, Greece, Hungary, Ireland, Luxembourg, Netherlands, Norway, Poland, Portugal, Spain, Sweden,and Switzerland.

[3]Singapore and Hong Kong, the latter being considered as a country for the purpose of this study.

[4]Global GDP is taken from the World Development Indicators database, as released in 2016.

1. **The World Bank World Development Indicators database** provides most of the data required to estimate the parameters for the model. All the data for absolute values are in current local currency units and converted to PPP using the conversion ratio for comparability.

   - **Nominal GDP** is formed as a sum of the two variables: *Gross value added at factor cost* and *Net taxes on products*, since the aggregate time series for the considered list of countries is not available. The only country for which GDP was considered directly is China, as GDP-at-factor-cost data are unavailable. GDP-at-factor-cost data are largely available for the entire considered period, except in the case of Argentina and Indonesia. It is important to mention that the missing data point for the Argentinian economy is the PPP conversion factor, not the GDP at factor cost or net taxes. Taking into account the volatility of the Argentinian peso, it is impossible to make any assumptions on the behavior of the conversion rate. The missing data points for the GDP at factor cost and net taxes, both here and later in this subsection, are thus filled in by assuming that the Argentinian share (e.g., GDP at factor cost) of the total 36 countries considered (in this case, the GDP at factor cost of 36 countries) remains stable. The data for Indonesia are available only from 2010. The missing data for 1991-2009 are thus projected through linear regression of the data available for the 2010-2014 time frame. Likewise, net tax data are missing for Indonesia and filled in using the same method described above for GDP at factor cost. Additionally, net tax data are missing for the USA in 1991-1996 and Hong Kong in 1991-1999. The missing data were also built using linear regression.
   - **GFCF** or *Gross fixed capital formation* is the variable that captures the level of investment in the economy and is defined by the World Bank as follows:"Gross fixed capital formation (formerly gross domestic fixed investment) includes land improvements (fences, ditches, drains, and so on); plant, machinery, and equipment purchases; and the construction of roads, railways, and the like, including schools, offices, hospitals, private residential dwellings, and commercial and industrial buildings. According to the 1993 SNA, net acquisitions of valuables are also considered capital formation". This variable is available for all the countries listed, except Argentina, for which the only available data point is for 2011 due to the lack of a conversion factor, as mentioned above. As for the earlier missing values, these were estimated using the overall ratio of Argentinian GFCF to the total GFCF for the 36 countries.

- **Household final consumption expenditure** (formerly private consumption) as defined by the World Bank is: "the market value of all goods and services, including durable products (such as cars, washing machines, and home computers), purchased by households. It excludes purchases of dwellings but includes imputed rent for owner-occupied dwellings. It also includes payments and fees to governments to obtain permits and licenses. Here, household consumption expenditure includes the expenditures of nonprofit institutions serving households, even when reported separately by the country." Household consumption in PPP dollars is not available directly for Argentina, so this was calculated via the available ratio of the Argentinian household consumption to the total household consumption of the 36 countries.
- **The employment rate** for the 36 countries considered was calculated as the percentage of employed population within the 15-64 age group. This was a two-step method: firstly, the number of people within the 15-64 age group was calculated from the percentage of the total population data and, secondly, the number of employed persons was calculated using data on the share of the employed population within the 15-64 age group.
- **GDP deflator (index)** is calculated as the ratio of the country's GDP in current prices to its GDP in constant prices. Both data sets are available through the World Bank World Development Indicators database for all 36 countries considered.

2. **The Penn World Table** provided data on capital stock and the share of employees' wages in the economy.

   - **Capital stock** data are already provided in PPP dollar units and available directly for all 36 countries.
   - **Employee compensation** is calculated from the share of labor compensation in GDP available through the World Penn Tables. The share of labor compensation in GDP is available for all 36 countries except Russia. The average of the respective shares for Kazakhstan, Ukraine, and Belarus is taken as a proxy to calculate employee compensation in Russia.

3. **The BIS Statistics Explorer** provides data on government, private and household debt. The missing data are filled in using the World Bank World Development Indicators database and the Economic Commission for Latin America and the Caribbean database, CEPALSTAT.

   - *Corporate debt* represents total credit to the non-financial corporate sector and is also defined as a percentage of GDP. The linear regression method was applied in order to fill in the missing data for: Brazil, Czech Republic, Luxembourg, Poland and Russia.

D.2 Temperature Dynamics

The temperature dynamics is based on the continuous-time, two-layer model described in Geoffroy et al. (2013):

$$\begin{cases} \overline{C}\dot{T} = F - (RF)T - \gamma^*(T - T_0), \\ C_0\dot{T}_0 = \gamma^*(T - T_0). \end{cases}$$

Nordhaus (1993) published what was to be his seminal model of greenhouse-gas emissions: the DICE model. In DICE, the two-layer model is used in a discrete formulation. In the present study, the parameters for the continuous-time model are calibrated based on the CO_2 trajectories reported in the technical report by Nordhaus and Sztorc (2013) p.30, and thus the global temperature change trajectories, p.31.

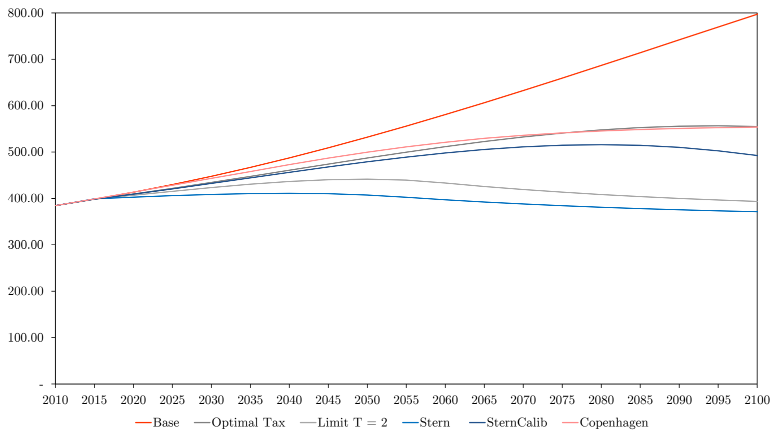

FIGURE D.1: CO_2 concentration in ppm under different scenarios. Source: Nordhaus and Sztorc (2013), p.30.

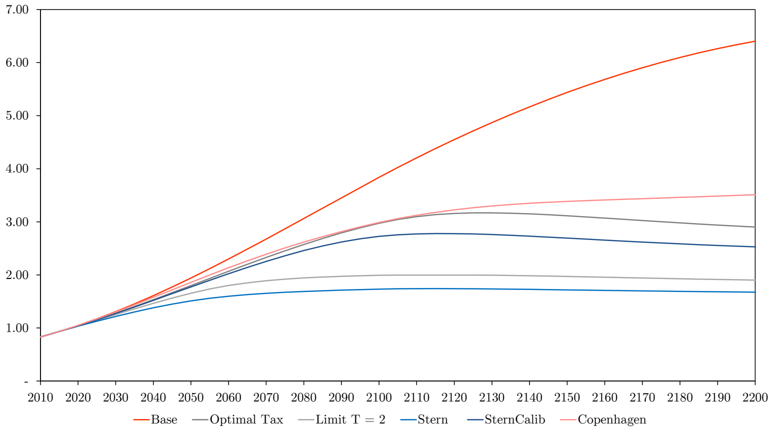

FIGURE D.2: Temperature change under different scenarios. Source: Nordhaus and Sztorc (2013), p.31.

To identify the two-layer model in its discrete version as described by Nordhaus and Sztorc (2013), the system is

$$\begin{cases} \dot{T} = \zeta_1\big(F - \zeta_2 T - \zeta_3(T - T_0)\big) \\ \dot{T_0} = \zeta_4(T - T_0). \end{cases}$$

One can identify the two models so that $\zeta_1 = \frac{1}{C}$, $\zeta_2 = RF$, $\zeta_3 = \gamma*$, and $\zeta_4 = \frac{\gamma*}{C_0}$. The equilibrium value $\zeta_2 := \frac{F_{2\times \mathrm{CO}_2}}{S} = 3.8/2.9$ is not calibrated according to the DICE trajectory as it represents an equilibrium value of the impact of a doubled CO_2 concentration in the atmosphere. In other words, the temperature anomaly is $S = 2.9$ if the CO_2 concentration is doubled in the atmosphere. While the endogenous radiative forcing is similar to Geoffroy et al. (2013), an additional feature is incorporated with an exogenous forcing so that:

$$F(t) = \frac{F_{2\times \mathrm{CO}_2}}{\log(2)} \log\left(\frac{C_{\mathrm{CO}_{2(t)}}}{C_{\mathrm{CO}_{2(t0)}}}\right) + F_{exo}(t).$$

Firstly, as usual, the parameter $F_{2\times CO_2} = 3.8$. Secondly, CO_2 is the major but not the sole cause of global warming. Future warming is also projected to come from other long-lived greenhouse gases, aerosols, and other factors. As in DICE, we assume that they are controlled exogenously following the process

$$\dot{F}_{exo} = \delta_{F_{exo}} F_{exo}\big(1 - \frac{F_{exo}}{0.7}\big).$$

In 2100, 0.7 W/m^2 is the estimated non-CO_2 forcing, and the value in 2010 is 0.25 W/m^2. In DICE 2013 code, this function is supposed to grow linearly until 2100 and then become constant at 0.7. To approximate this behavior smoothly, the parameter $\delta_{F_{exo}}$ will be 0.25, and the yearly starting values will be as indicated in Table D.1,

Year	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019
Value	0.25	0.26	0.27	0.28	0.29	0.30	0.31	0.32	0.33	0.34 .

TABLE D.1: Starting values for the exogenous forcing dynamics.

The Nordhaus DICE model is calibrated for a five-year time mesh, whereas our calibration is designed for one year. To ensure consistency with our time mesh, we infer the parameters $(\zeta_1, \zeta_3, \zeta_4)$ to ensure a minimal distance between their trajectories and those displayed in Figure D.2.[5]

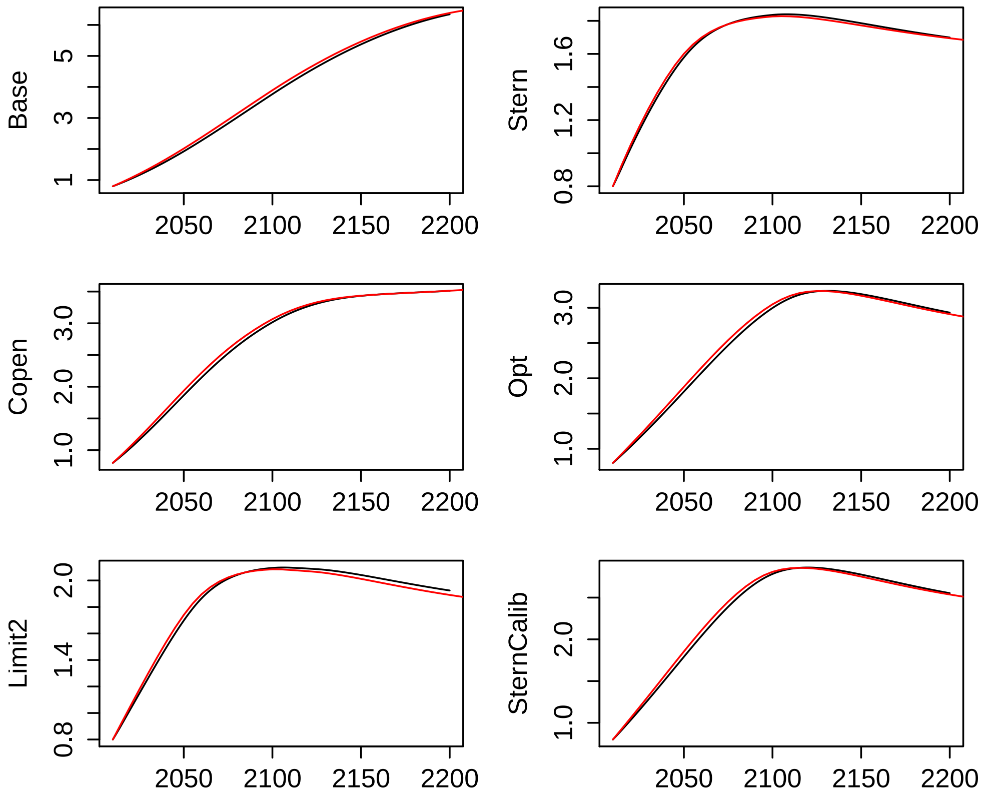

FIGURE D.3: Fitting of the parameters for the continuous system to the discrete counterpart. In black, the simulated values with the continuous-time system. In red, the DICE temperature increase.

[5]The methodology involves finding the set of parameters that produces trajectories as close as possible to the values generated by DICE. For the sake of clarity, the values found will minimize the Euclidean distance via implementation of a BFGS algorithm.

Parameter	Value
ζ_1	0.019716867
ζ_3	0.125815568
ζ_4	0.007654311

TABLE D.2: Estimated values of the zetas for the continuous-time system.

D.3 The Demographic Scenario

The working-age population is assumed to grow according to some logistic function,

$$\frac{\dot{N}}{N} = q\big(1 - \frac{N}{M}\big).$$

In order to infer the parameters (q, the speed rate, and, M, the upper limit), we minimize the distance between the simulated process and the United Nations median fertility scenario using a BFGS algorithm.[6]

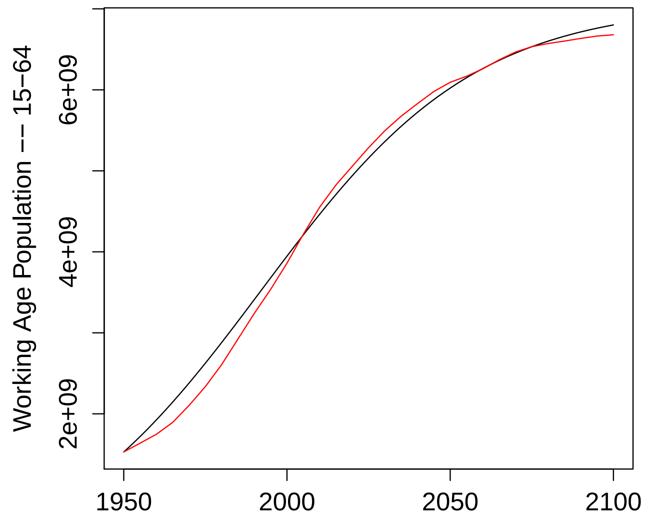

FIGURE D.4: Fitting of the simulated scenario to the 15-64 age group scenario.

Sources: United Nations.

Figure D.4 shows the fitting of the logistic function (in black) to the United Nations time series (in red). The parameters are displayed in Table D.3.

[6]The time series for the working-age population is the median fertility scenario for the 15-64 age group.

Parameter	**Value**
q	0.0305
M	7,055,925,493

Table D.3: Calibrated parameters.

D.4 CO_2 Dynamics

The measures of the carbon cycle are in gigatons of carbon (hereafter GtC) for CO_2 concentration, and emissions are measured in gigatons of CO_2 (hereafter $GtCO_2$). For the sake of clarity, Table D.4 shows the conversion factor of the different carbon cycle units. It displays three units: GtC, $GtCO_2$, and part per million (hereafter ppm).

From \ To	GtC	$GtCO_2$	ppm
1 GtC	1	3.664	1/2.13
1 $GtCO_2$	1/3.664	1	1/7.81
1 ppm	2.13	7.81	1

Table D.4: Conversion table for carbon cycle metrics.

According to Nordhaus and Sztorc (2013), the CO_2 accumulates in three layers: (i) the atmosphere; (ii) a mixing reservoir in the upper oceans and the biosphere; and (iii) the deep ocean. This mechanism is represented by the system

$$\begin{pmatrix} \dot{CO_2}^{AT} \\ \dot{CO_2}^{UP} \\ \dot{CO_2}^{LO} \end{pmatrix} = \begin{pmatrix} E \\ 0 \\ 0 \end{pmatrix} + \underbrace{\begin{pmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{pmatrix}}_{:=\Phi} \begin{pmatrix} CO_2^{AT} \\ CO_2^{UP} \\ CO_2^{LO} \end{pmatrix},$$

The coefficients of matrix Φ are linked together as follows:

$$\Phi = \begin{pmatrix} -b_{12} & b_{12}\frac{C_{AT_{eq}}}{C_{UP_{eq}}} & 0 \\ b_{12} & -b_{12}\frac{C_{AT_{eq}}}{C_{UP_{eq}}} - b_{23} & b_{23}\frac{C_{UP_{eq}}}{C_{LO_{eq}}} \\ 0 & b_{23} & -b_{23}\frac{C_{UP_{eq}}}{C_{LO_{eq}}} \end{pmatrix},$$

where b_{ij} is the flow of CO_2 between the layers i and j, and $C_{i_{eq}}$ is the equilibrium concentration of the layer i, constant for the modeling.

A discrete version of this system described by Nordhaus and Sztorc (2013) is more intuitive in terms of CO_2 flows into the atmospheric layer and flows between the layers:

$$\begin{pmatrix} CO_2^{AT}(t) \\ CO_2^{UP}(t) \\ CO_2^{LO}(t) \end{pmatrix} = \begin{pmatrix} E(t) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1-b_{12} & b_{12}\frac{C_{AT_{eq}}}{C_{UP_{eq}}} & 0 \\ b_{12} & 1-b_{12}\frac{C_{AT_{eq}}}{C_{UP_{eq}}}-b_{23} & b_{23}\frac{C_{UP_{eq}}}{C_{LO_{eq}}} \\ 0 & b_{23} & 1-b_{23}\frac{C_{UP_{eq}}}{C_{LO_{eq}}} \end{pmatrix} \begin{pmatrix} CO_2^{AT}(t-1) \\ CO_2^{UP}(t-1) \\ CO_2^{LO}(t-1) \end{pmatrix}.$$

The DICE model is calibrated for a five-year time mesh, whereas our calibration is designed for one year. To have parameters consistent with our time mesh, we calibrate b_{12} and b_{23} so that the continuous trajectories obtained with these parameters minimize the distance from the discrete counterpart of Nordhaus' DICE model, as displayed in Figure D.5.

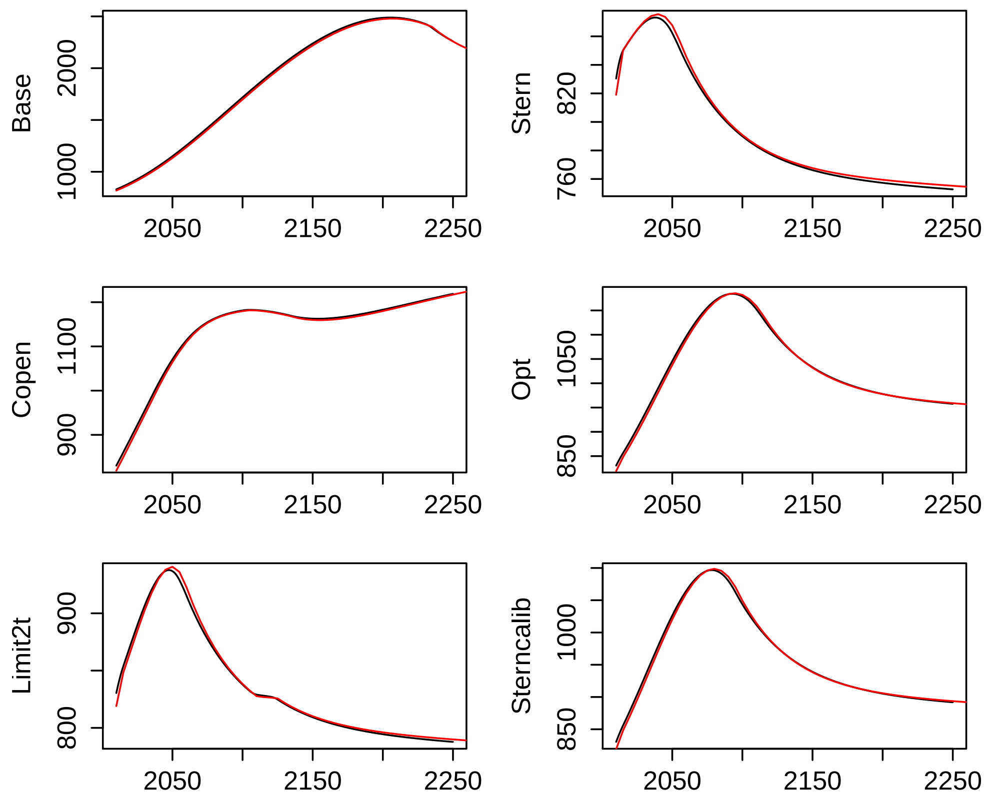

FIGURE D.5: Fitting of the parameters for the continuous system to the discrete counterpart. In black, the simulated values with the continuous-time system. In red, the DICE CO_2 accumulation in the atmosphere.

In other words, our estimates minimize the Euclidean distance between our continuous simulation and the trajectories interpolated from the DICE model, which served as a reference, meaning,

$$(\hat{b}_{12}, \hat{b}_{23}) = \text{argmin}_{(b_{12},b_{23})} d(obs, sim(b_{12}, b_{23}, C_{AT_{eq}}, C_{UP_{eq}}, C_{LO_{eq}})),$$

where obs is the vector of observation of the DICE model over the 2010-2250 period, $sim(b_{12}, b_{23}, C_{AT_{eq}}, C_{AT_{eq}}, C_{AT_{eq}})$ is the corresponding vector taken from the simulation using the couple (b_{12}, b_{23}) as the argument of the estimation and the constants $(C_{AT_{eq}}, C_{UP_{eq}},$
$C_{LO_{eq}})$, and d is the Euclidean distance. Using a BFGS algorithm, we obtain the estimated values given in Table D.5.

Parameter	Value
b_{12}	0.01727393
b_{23}	0.00050000

TABLE D.5: Calibrated parameters.

D.5 Estimation of the Macroeconomic Dynamics

The estimations that follow are based on the data defined in Appendix D.1.

D.5.1 The Capital-to-Output ratio, ν

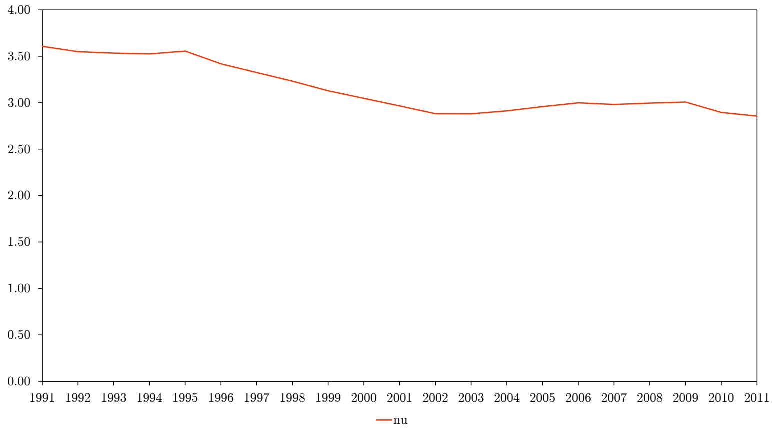

FIGURE D.6: Capital-to-output ratio from 1990 to 2011.

Sources: World Bank, Penn.

Figure D.6 shows the capital-to-output ratio, ν, over time. Note that ν is not constant, although roughly stable after 2000 at an average of 2.9485. This value is in line with previous findings or assumptions (see Appendix C in Feenstra et al. (2015) or the AMECO database, which assume an initial value of capital-to-output ratio of 3 in 1960).

D.5.2 Labor Productivity, a_t

For the different labor productivity scenarios, the estimation below helped us develop our baseline scenario. Figure D.7 shows the growth rate over time of labor productivity. For its inference, we do not use the OLS regression as suggested in Grasselli and Maheshwari (2016),

$$\log(a_t) = \log(a_0) + \beta t + \varepsilon_t,$$

because the low growth rates in the early 1990s would strongly bias the estimate of the slope β. In order to get a better approximation, we thus infer α so that it equals the mean value of the time series, i.e., 0.0226.

FIGURE D.7: The labor productivity growth rate from 1990 to 2011.

Sources: World Bank, Penn.

D.5.3 The Short-Term Phillips Curve

In order to calibrate an aggregate short-term Phillips curve, we estimate the model

$$\frac{\dot{w}}{w} = \phi_0 + \phi_1 \lambda,$$

where w denotes the nominal wage rate, λ the employment rate, ϕ_0 the intercept, and ϕ_1 the slope of the aggregate function.

For each country, the data are taken from the sources presented in Appendix D.1. The employment rate, λ, is taken as given, while the wage per capita, w, is computed as the total wage bill divided by the labor force.

Due to the very low volatility of the data at an aggregate level (the aggregate employment rate, λ, ranges from 0.69 to 0.73),[7] no significant econometric estimate could be made at this level of aggregation. As a result, a panel data regression analysis was used for the 36 selected countries to improve the inference. In order to cope with temporal autocorrelation and omitted variables, the "first-difference" model was selected to estimate the slope of the function. The estimated model is as follows:

$$\left(\frac{\dot{w}}{w}\right)_{it} \sim \lambda_{it},$$

where i denotes the country and t the year of estimation. Note that we constrain the intercept to be zero.

Using the *plm* package (free *R* software), we obtain significant results, presented in Table D.6.

Coefficient	**Estimate**	**Std Error**	**t-value**	**Pr(> |t|)**	**Observations**
ϕ_1	1.08519	0.29034	3.7376	0.0002013***	720

Table D.6: First-difference panel regression of the short-term Phillips curve over the period 1991-2010.

Sources: World Bank, Penn.

Next, given the estimated slope for the aggregate wage function, we calibrate the intercept in order to match the empirical first moment of the sample,

$$\mathbb{E}\left(\frac{\dot{w}}{w}\right) = \phi_0 + \phi_1 \mathbb{E}(\lambda).$$

Finally, we obtain the following relation for the short-term Phillips curve:

$$\frac{\dot{w}}{w} = -0.735026 + 1.08519\lambda.$$

Figure D.8 presents the real and fitted wage curves and highlights the robustness of our estimation. The Kolmogorov-Smirnov test rejects the null hypothesis of Gaussian residuals. However, the qualitative results are displayed in Figure D.8, the estimation of a model with a non-Gaussian distribution being left for further research.

[7] Strongly driven by India and China in terms of population size.

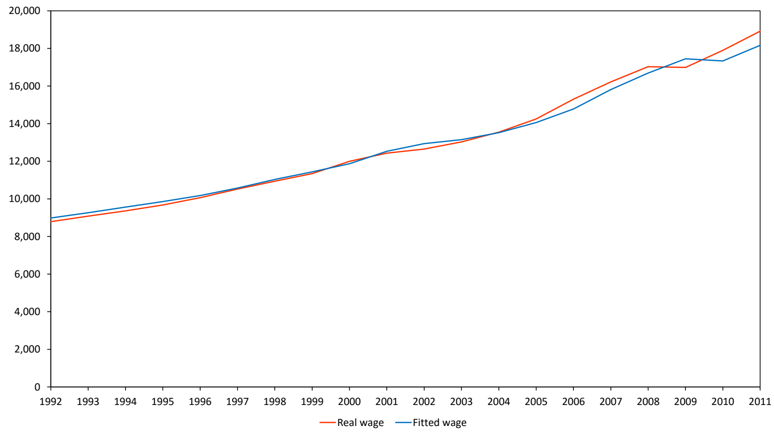

FIGURE D.8: Observed and fitted wage curve over the period 1992-2010

Sources: World Bank, Penn.

Note that we conducted the same panel regression analysis taking both the employment rate λ and the inflation i of each country in the sample as explanatory variables. Inflation was computed as the growth of the GDP deflator presented in Appendix D.1. The p-value of 0.77 for the inflation parameter shows that the null hypothesis of this parameter being zero is not rejected. After this model selection process, we thus only consider the specification of the short-term Phillips curve with λ as the sole explanatory variable for our calibration, and we assume complete monetary illusion.[8]

Coefficient	**Estimate**	**Std Error**	**t-value**	**Pr(> |t|)**	**Observations**
ϕ_λ	1.07991	0.29111	3.7096	0.0002245***	720
ϕ_i	0.00062	0.00217	0.2891	0.7725579	720

TABLE D.7: First-difference panel regression of the short-term Phillips curve with λ and i as explanatory variables over the period 1991-2010. Sources: World Bank, Penn.

As a caveat, we should point out that residuals are autocorrelated.

D.5.4 The Investment Function

In order to calibrate the aggregate investment function, we estimate the following model:

$$\frac{I}{Y} = i_0 + i_1\pi,$$

[8]It is worth mentioning that the constant parameter incorporates the information that wages reset automatically, given a constant inflation rate.

where I is real investment, Y is real output, π is the profit share, i_0 is the intercept, and i_1 is the slope of the aggregate function.

The data are drawn from the sources previously mentioned in Appendix D.1. Output Y and investment I are taken as given (we use GDP at factor cost as government is not modeled). We must point out that, due to data availability issues, we use these nominal frameworks and assume constant price relativity for the identification. The profit share, π, is computed as the ratio of GDP at factor cost net of wages (gross profit) and net of interest payments on debt. As we have information only about the level of debt for the 36 countries selected, we assume a constant interest rate of 3% for our calculations.

Using the free *R* software, we estimate the following model on aggregate series of the 36 selected countries with an ordinary least square (OLS) regression:

$$\left(\frac{I}{Y}\right)_t \sim \pi_t.$$

Table D.8 presents the significant results of the estimation. A Kolmogorov-Smirnov test applied to the normality of the residuals shows a p-value of 0.4448. Hence, with a 95% confidence band, this test does not reject the null hypothesis of the normality of the errors.

Coefficient	**Estimate**	**Std. Error**	**t-value**	**Pr(> |t|)**	**Observations**
i_0	0.04260	0.01288	3.308	0.0037**	36
i_1	0.64153	0.03773	17.002	5.96e-13***	36

TABLE D.8: OLS regression of the investment function over the period 1991-2011

Sources: World Bank, Penn.

Finally, we obtain the following relation:

$$\frac{I}{Y} = 0.04260 + 0.64153\pi.$$

Figure D.9 presents the real and fitted wage curves and highlights the robust behavior of the estimation. Note that no issues regarding the residuals were identified for this estimation. Indeed, a Kolmogorov-Smirnov test applied to the normality of the residuals shows a p-value of 0.4448. Hence, this test does not reject the null hypothesis of the normality of the errors.

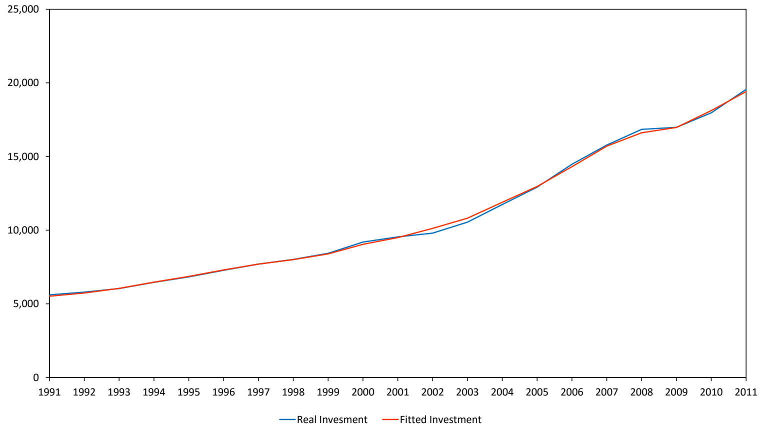

FIGURE D.9: Observed and fitted investment curve over the period. 1992-2010.

Sources: World Bank, Penn.

D.5.5 Debt Accumulation

As shown by Figure D.10, a mismatch exists between observed debt accumulation and observed investment net of profits. Indeed, firms may become indebted for purposes other than investing, as for instance, for speculative reasons (note the singular pattern during the financial bubble over the period 2000-2007), for distributing dividends or paying taxes. By way of illustration, the US monetary-dividend-to-GDP ratio presented below appears to be highly correlated with the debt-variation-to-GDP ratio and its level in line with the corresponding estimated parameter presented below.

In order to reconcile the model with the data, a corrective term was added to the debt accumulation cinetic: $\dot{D} = \text{div } pY + pI - \Pi$, where div pY stands for the "grey debt". Within the present set-up, the most natural interpretation of this addition is in terms of dividends paid by firms to their shareholders.[9]

[9] Once a public sector will be made explicit in a subsequent paper, other interpretations of the "grey debt" (e.g., in terms of taxes paid by firms) will be available.

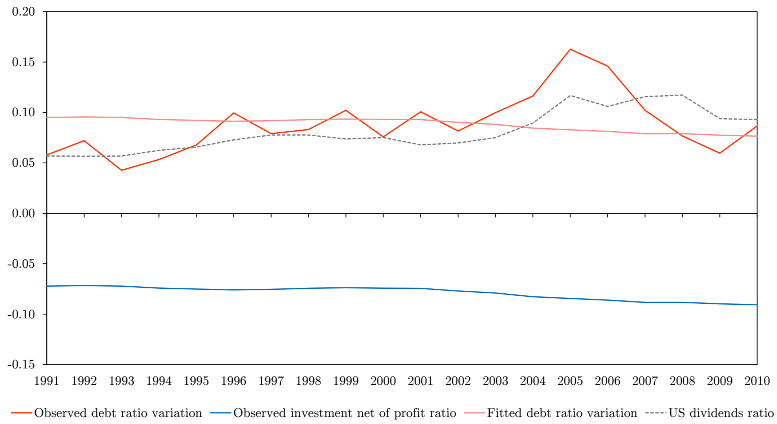

FIGURE D.10: Observed aggregate debt variation (red curve) and investments net of profit (blue curve), fitted aggregate debt variation (light red curve) and US monetary dividend (gray dotted curve) over the period 1991-2010 (all as ratios of GDP). Sources: World Bank, Penn, BEA.

The constant, div, was calibrated to reconcile the first moments of the debt accumulation function with respect to the data, that is,

$$\mathbb{E}\left(\frac{\dot{D}}{pY}\right) = \text{div} + \mathbb{E}\left(\frac{pI}{pY} - \pi\right).$$

We used the fitted value of investment for the calculation in order to obtain the best fit for our model, and found a value of 0.1672287 for div.

D.5.6 The Depreciation Rate of Capital, δ

Due to the lack of identification, the depreciation rate of capital is often calibrated as an educated guess (see Smets and Wouters (2007) among others). Here, one of the drivers of the real output growth rate is the law of motion of capital,

$$\frac{\dot{K}}{K} = \frac{\kappa(\pi)}{\nu} - \delta.$$

Having previously calibrated $\kappa(\pi)$ and ν, we now calibrate δ so that the right-hand side of the equation matches the empiricallt observed real growth rate of GDP:

$$\mathbb{E}\left[\frac{\kappa(\pi)}{\nu} - \frac{\dot{Y}}{Y}\right] = \hat{\delta}.$$

By doing so, we find $\hat{\delta} = 0.0645$.

D.5.7 The Price Dynamics

The price dynamics model is represented as follows:

$$\frac{\dot{p}}{p} = \eta_p(m\omega - 1) + c,$$

where $\eta_p, c > 0$ and $m \geq 1$. In this relationship, we identify c to some long-term inflation trend and $\eta_p(m\omega - 1)$ to some frictional inflation. In other words, and for the purpose of the estimation, we assume that $\mathbb{E}\left(\eta_p(m\omega - 1)\right) = 0$, so that $c = \mathbb{E}\left(\frac{\dot{p}}{p}\right)$. Using the first relation, we calibrate m as $m = 1/\mathbb{E}(\omega)$, and obtain c from the second relation. The parameter η_p being set, we focus on the temporary component of inflation and calibrate η_p such that the deviation of short-term inflation with respect to c does not exceed 0.05 in absolute terms, assuming the wage share ω to be contained in a reasonable range, $[0.24, 1]$.

Table D.9 summarizes the calibration retained for the numerical simulations. The data are taken from the previously presented sources and the estimation was carried out for the period 1991-2011.

Parameter	**Value**
c	0.030322
m	1.609972
η_p	0.0819709341

Table D.9: Calibration of the price dynamics parameters.

Sources: World Bank

D.6 Addtional Scenarios

D.6.1 The Gordon Case

Figure D.11 shows the paths of the key macroeconomic and climate variables in the Gordon labor productivity growth. Despite the fact that this scenario shows, as expected, similar patterns as previously, it is worth mentioning that the reduction of 16.4% of the CO_2 accumulation in 2100 and thus a difference of atmospheric temperature of 3.7%.

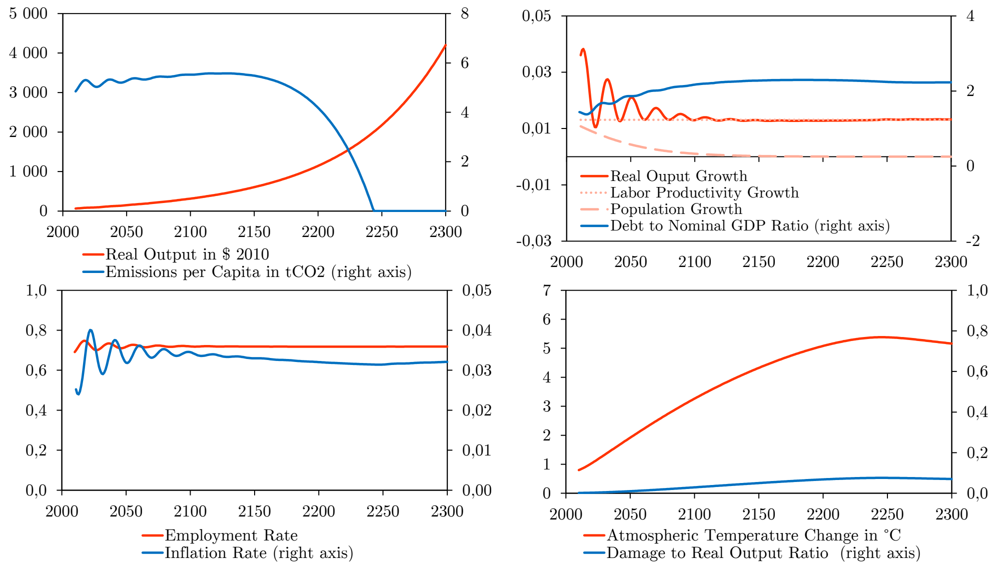

FIGURE D.11: Trajectories of the main simulation outputs in the Gordon labor productivity growth and Nordhaus damage function case.

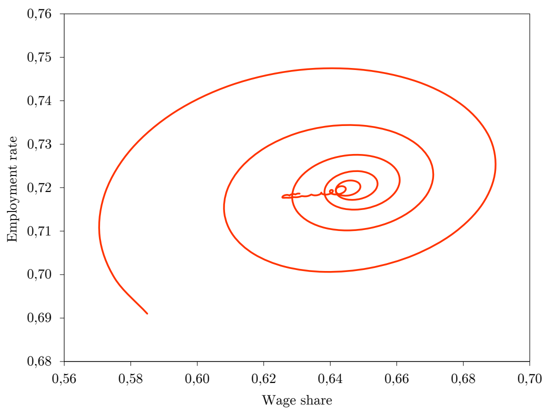

FIGURE D.12: Phase diagram of employment rate versus the wage share in the Gordon labor productivity growth and Nordhaus damage function case.

D.6.2 The Nordhaus Case

For the sake of comparison with the baseline scenario of DICE, Figure D.13 presents the trajectories of the main macroeconomic and climate variables using another deterministic exponential labor productivity that closely follows the growth rate of the economy in DICE. This is reflected by a calibration of a labor productivity increase of 1.5% a year. At the meantime, this scenario aims to perform a sensitivity analysis on the labor productivity growth as one of the main driver of economic growth.

As expected, this deterministic exponential labor productivity, lower than the previous case, induces a muted global behavior as shown in Figure D.13 so that the real output level, the temperature and the damages are lowered. In terms of being specific the temperature anomaly is 3.39°C in 2100 and the CO_2 emissions become 74.69 Gt CO_2 at the same date.

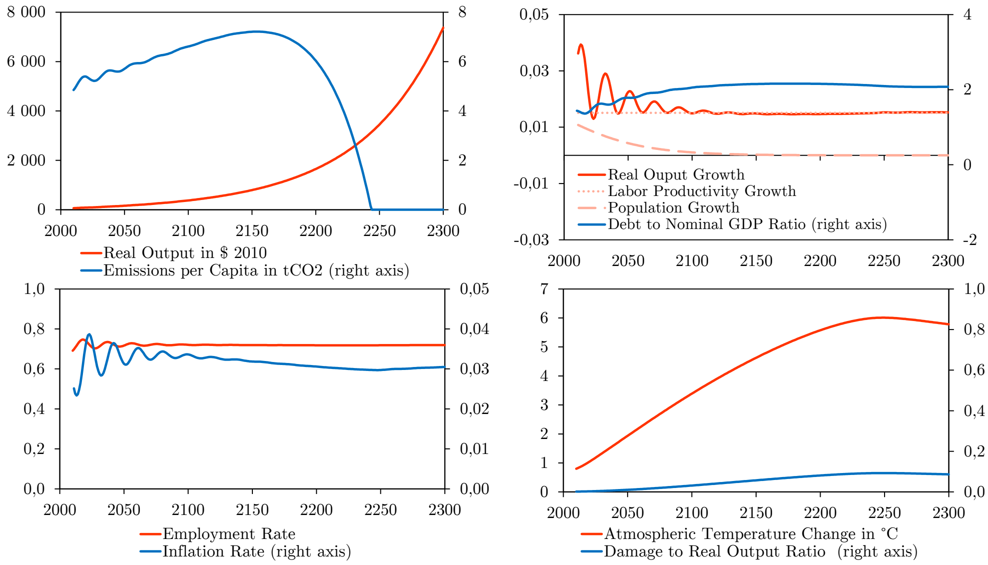

FIGURE D.13: Trajectories of the main simulation outputs in the Nordhaus labor productivity growth and Nordhaus damage function case.

Besides that most of the variables are muted, few differences are worth mentioning: i) as shown in Figure D.14, the deviation from the long-run equilibrium induced by climate change observed in the BAU scenario shows a modest deviation with respect to its counterpart, this is the consequence of a lower emission trajectory, hence a muted impact of damages in the real production; ii) the debt-to-output ratio shows similar variations at a higher level (greater than 2 in 2300, whereas the BAU scenario showed a level lower that 1.5 at the same date). The major effect resulting from a lower productivity growth in this Lodka-Volterra logic is a higher wage share, thus profit losses is compensated by a higher indebtedness.

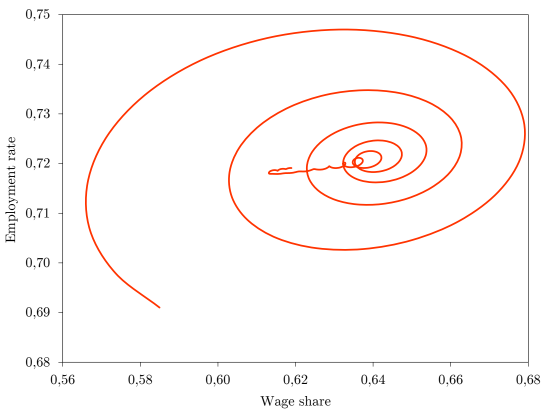

FIGURE D.14: Phase diagram of employment rate versus the wage share in the Nordhaus labor productivity growth and Nordhaus damage function case.

D.6.3 The Gordon - Weitzman Case

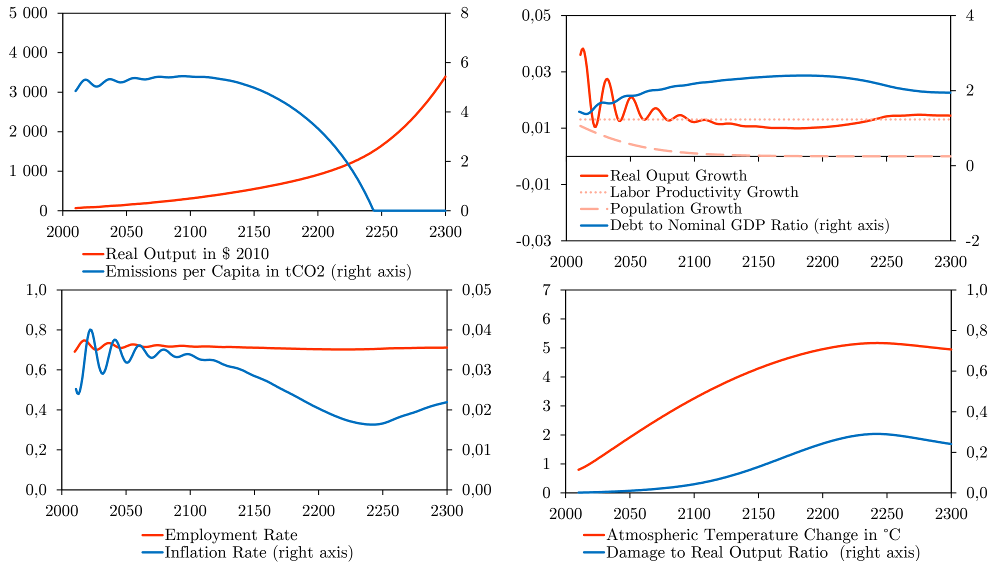

FIGURE D.15: Trajectories of the main simulation outputs in the Gordon labor productivity growth and Weitzman damage function case.

Figure D.15 presents the paths of the main macroeconomic and climate variables in the Gordon labor productivity growth and the Weitzman damage function case. This scenario represents a median case with a constant, but limited, growth of labor productivity and an intermediate damage function. We do not observe any collapse while the temperature change goes up to $5°C$ in the atmosphere layer in 2300. Indeed, even if they are quite high (30% of the real output around 2230), the damages do not sufficiently, in regard to the previous scenario, deteriorate the real output and, therefore, to generate a collapse and the economy can complete the energy shift at the cost of a rise of the debt ratio and a fall of the wage share.[10]

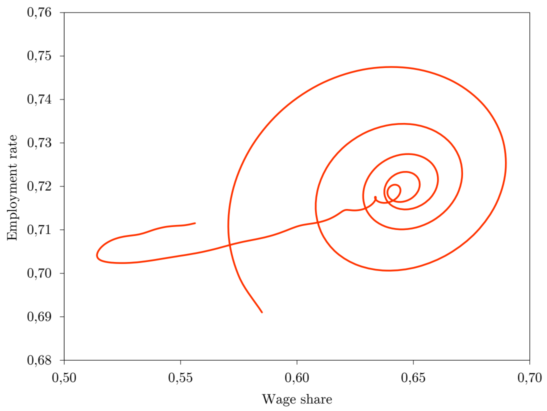

FIGURE D.16: Phase diagram of employment rate versus the wage share in the Gordon labor productivity growth and Weitzman damage function case.

[10]This result is similar to the case with a damage function à la Nordhaus.

D.7 The Abatement Costs

We follow the calibration of Nordhaus and Sztorc (2013) for (θ_1, θ_2) with $\theta_1 = \frac{\sigma p_{BS}}{1000\theta_2}$ and $\theta_2 = 2.8$.

The calibration of these exogenous trajectories is inspired by the baseline case of sNordhaus and Sztorc (2013) as a weakly coercive constraint for the economy.

D.8 Elements of Backtesting

This section provides some elements of back-testing of the macroeconomic module. The main parameters of the simulation are initialized with their 1991 value as presented in Table D.10.[11] The horizon of the simulation is set at 20 years and the climate module is disabled due to its negligible impact over the period (less than 1% of damages in 2010 as previously exposed). We assume an exponential technological progress for this period at a growth rate of 0.015 borrowed from Nordhaus, which stands as a standard value in the academic literature and a median value for our scenarios.

Parameter	Y_{1991}	N_{1991}	ω_{1991}	λ_{1991}	d_{1991}	p_{1991}	α
Value	22.2493	3.2998	0.6452	0.7280	1.1544	1	0.015

Table D.10: Main macroeconomic parameters of the back-testing case.

We finally compare the obtained trajectories with the observed timeseries over the period 1991-2011 as shown in Figure D.17. The presented curves qualitatively assess the robust behavior of this modeling over a short time-period. On the medium and long term, the order of magnitude of the observed timeseries are respected in spite of major macroeconomic shifts and turmoils such as the Internet bubble of 2000 or the financial crisis of 2008.

[11] As a caveat, let us denote the parameters of the working age population dynamics have also been calibrated in order to fit the observed demography over the back-testing period 1991-2011. The speed rate q has been reduced from 0.03805 to 0.03803. The upper-limit M has been maintained to its previous calibration value of $7,055,025,493$. No other macroeconomic dynamics have been subject to any other change.

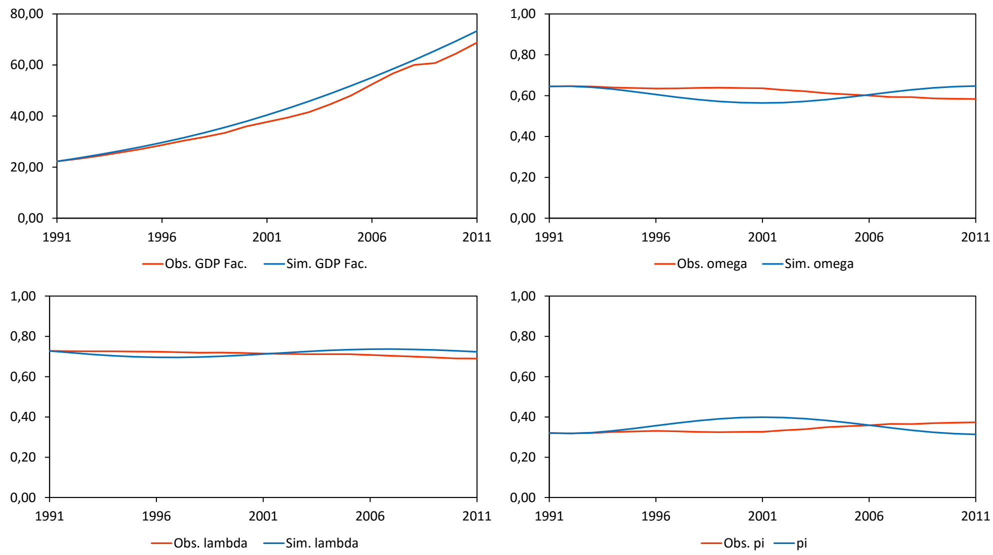

FIGURE D.17: Back-testing of the macroeconomic module over the period 1991-2011 in an exponential technological progress without climate back-loop case.

D.9 Shifting long-term equilibra

The model presented boils down to a 16-dimensional dynamical system. Once the energy shift is fully performed, there are no longer additional emissions and the climate module will converge to a unique stable equilibrium characterized by a constant positive mean atmospheric temperature deviation T_{eq}.[12] The economic and climate modules are then autonomous. As a result, an analysis of the macroeconomic module at the climate equilibrium can be performed. The system below presents this module in its very general form (including damage on output, capital and labor productivity). Doing so, we derive the impact of climate change on the equilibrium shape with every damage channel.

D.9.0.1 Differential system

The system of differential equations constituting the macroeconomic module breaks down as follows

[12] It is worth mentioning that the only exogenous remaining term of the climate module is F_{exo}. This exogenous forcing follows a sigmoid dynamics, and reaches its upper limit at equilibrium.

$$\begin{cases} \dot{\omega} & = \omega \left[\Phi(\lambda) - (1-\gamma) i(\omega) - \frac{\dot{a}}{a} + \frac{\dot{D}}{1-D} \right] \\ \dot{\lambda} & = \lambda \left[\frac{\dot{Y}}{Y} - \frac{\dot{a}}{a} - \frac{\dot{N}}{N} + \frac{\dot{D}}{1-D} \right] \\ \dot{d} & = d \left[r - \left(\frac{\dot{Y}}{Y} + i(\omega) \right) \right] + \kappa(\pi) + Div - (1-\omega) \\ \frac{\dot{N}}{N} & = q \left(1 - \frac{N}{P^N} \right), \end{cases}$$

with the additional variable depending on parameters and reduced variables

$$\begin{aligned} \pi &= 1 - \omega - rd, \\ \frac{\dot{Y}}{Y} &= \frac{1}{\nu}(\kappa(\pi) - \mu G)(1-D) - \delta - \frac{\dot{D}}{1-D}, \\ &= \frac{\dot{a}}{a} + \frac{\dot{\lambda}}{\lambda} + \frac{\dot{N}}{N} - \frac{\dot{D}}{1-D}, \\ \frac{\dot{a}}{a} &= g_a(T_{eq}), \\ i(\omega) &= \eta_p(m\omega - 1) + c. \end{aligned}$$

At the climate equilibrium, environmental damages are constant and we can consider $G = 0$.[13] The complexity added by the inflation makes impossible to derive explicit value of the equilibrium. However, as shown by Grasselli and Nguyen-Huu (2015), if the price dynamics of course alter the dynamics behavior and calibration of the system, it does not radically change the typology of equilibria. That is why we will assume for the rest of the mathematical analysis that there is no inflation ($i(\omega) = 0$) that will allow to shed more intuitive insight on climate change impact on the economy, and relax this assumption for the numerical analysis performed with the calibrated model. Within those assumptions, the system becomes

[13]Due to the price of the backstop technology, abatement costs exponentially converges toward zero and can be considered null as the energy shift is performed.

$$\begin{cases} \dot{\omega} & = \omega \left[\Phi(\lambda) - g_a(T_{eq}) \right] \\ \\ \dot{\lambda} & = \lambda \left[\frac{\dot{Y}}{Y} - g_a(T_{eq}) - \frac{\dot{N}}{N} \right] \\ \\ \dot{d} & = d \left[r - \frac{\dot{Y}}{Y} \right] + \kappa(\pi) + Div - (1-\omega) \\ \\ \frac{\dot{N}}{N} & = q \left(1 - \frac{N}{P^N} \right), \end{cases}$$

with the additional relations

$$\begin{aligned} \pi &= 1 - \omega - rd, \\ \frac{\dot{Y}}{Y} &= \frac{\kappa(\pi)}{\nu}(1 - D(T_{eq})) - \delta, \\ \frac{\dot{a}}{a} &= g_a(T_{eq}). \end{aligned}$$

D.9.1 Analysis of the desirable equilibrium

The latter system admits a desirable equilibrium[14] in which the economy reaches a steady-state growth path at the pace

$$\begin{aligned} g_{Y_{eq}} &:= \left. \frac{\dot{Y}}{Y} \right|_{eq}, \\ &= g_{a_{eq}}. \end{aligned}$$

At the equilibrium, the profit rate is defined by the following relation

$$\pi_{eq} = \kappa^{-1} \left(\nu \frac{(g_{Y_{eq}} + \delta)}{1 - D} \right),$$

[14] A desirable equilibrium logically require non zero employment rate and wage share as well as finite level of private debt.

Then, the shape of this equilibrium is given by the set of the following equations

$$\begin{cases} \omega_{eq} & = 1 - \pi_{eq} - rd_{eq}, \\ \\ \lambda_{eq} & = \Phi^{-1}(g_{Y_{eq}}), \\ \\ d_{eq} & = \frac{\kappa(\pi_{eq}) + Div - \pi_{eq}}{g_{Y_{eq}}}, \\ \\ N_{eq} & = P^N. \end{cases}$$

First, the equilibrium employment rate is immediately deduced from its steady-state growth rate. This is a direct consequence from the short term Phillips curve, and if the labor productivity falls due to global warming, so will do the real growth rate and the employment rate.

In addition, global warming increases the profit rate through the environmental damages allocated to output, while its effect through labor productivity will depend upon the chosen functional form. Assuming this relation relies on the contribution of Burke et al. (2015) (that is a quadratic form), we see that small temperature deviation will lead to higher productivity growth and thus profit share while more severe will consequently decrease both, all factors remain constant. However, at the equilibrium, in order to stabilize the employment rate, and to a certain extend growth, firms should invest more in order to compensate for either the losses on output or capital stock.

As a result, the economy is compatible with the Minsky's insights of financial instability, the debt ratio will consequently rise laying increased repayment burden on the economy.

D.9.2 Destabilizing climate change

Turning to to the stability analysis of the desirable equilibrium, the Jacobian matrix at this equilibrium reads

$$M(\omega_{eq}, \lambda_{eq}, d_{eq}, N_{eq}) \quad := \quad \begin{bmatrix} 0 & M_{12} & 0 & 0 \\ -M_{21} & 0 & -rM_{21} & M_{24} \\ M_{31} & 0 & M_{33} & 0 \\ 0 & 0 & 0 & M_{44} \end{bmatrix},$$

where $M_{ij}, (i,j) \in \{1,2,3\}$ are some coefficients defined such that

$$
\begin{aligned}
M_{12} &:= \omega\Phi'(\lambda) > 0, \\
M_{21} &:= \frac{\lambda_{eq}}{\nu}\kappa'(\pi_{eq})\big(1 - D_{eq}\big) > 0, \\
M_{24} &:= \lambda_{eq}\frac{q}{P^N}, \\
M_{31} &:= \left(\frac{d_{eq}}{\nu}(1 - D_{eq}) - 1\right)\kappa'(\pi_{eq}) - \Delta'(\pi_{eq}) + 1, \\
M_{33} &:= rM_{31} - g_{Y_{eq}}, \\
M_{44} &:= -q.
\end{aligned}
$$

The characteristic polynomial $\chi_M(\cdot)$ of the Jacobian matrix at the good equilibrium writes,

$$
\begin{aligned}
\chi_M(\epsilon) &= (\epsilon + q)\big[\epsilon^3 + (g_{Y_{eq}} - rM_{31})\epsilon^2 + ... \\
&\quad ... + M_{12}M_{21}\epsilon + g_{Y_{eq}}M_{12}M_{21}\big].
\end{aligned}
$$

The first root, $\epsilon = -q$, of the polynomial $\chi_M(\cdot)$ being obviously negative, the stability of the desirable equilibrium is given by the sign of the root of its factored polynomial of order 3. The latter is similar to the characteristic polynomial found by Grasselli and Lima (2012) for the study of the Goodwin-Keen model.

According to the Routh-Hurwitz criterion, a necessary and sufficient condition for the root of this polynomial to have a negative and non-null real part is

1. $g_{Y_{eq}} > rM_{31}$,
2. $(g_{Y_{eq}} - rM_{31})M_{12}M_{21} > g_{Y_{eq}}M_{12}M_{21}$ which is equivalent to $rM_31 < 0$ by positivity of M_{12} and M_{21}.

It is worth mentioning that, due to the quadratic specification for $g_a(T_{eq}) = g_{Y_{eq}}$, there is no reason to preclude $g_{Y_{eq}}$ in the positive domain as global warming may decrease the labor productivity. However, knowing that a negative growth rate of output would drive the economy out of the basin of attraction of the good equilibrium, we can exclude this situation that does not belong to its basin of attraction.

As a result, the stability necessary and sufficient condition of the good equilibrium boils down to $rM_{31} < 0$, that is:

$$r\left[\left(\frac{d_{eq}}{\nu}(1-D_{eq})-1\right)\kappa'(\pi_{eq})+1\right]<0$$

This condition can be numerically evaluated with the calibrated model in order to infer the impact of climate change on the necessary and sufficient condition of stability for the desirable equilibrium.

Assuming $r > 0$, and using the equilibrium definition of the debt ratio, a necessary condition of stability can be provided with

$$\pi_{eq} > \frac{\nu\delta}{1-D_{eq}} + Div.$$

As the equilibrium mean atmospheric temperature deviation increases, this expression makes explicit the destabilizing impact of global warming as the lower bound of the equilibrium profit rate rises while the latter must be remain below 1.

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